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The moduli space and M(atrix) theory of 9D $\mathcal{N} = 1$ backgrounds of M/string theory

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Abstract: We discuss the moduli space of nine dimensional $\mathcal{N} = 1$ supersymmetric compactifications of M theory / string theory with reduced rank (rank 10 or rank 2), exhibiting how all the different theories (including M theory compactified on a Klein bottle and on a Möbius strip, the Dabholkar-Park background, CHL strings and asymmetric orbifolds of type II strings on a circle) fit together, and what are the weakly coupled descriptions in different regions of the moduli space. We argue that there are two disconnected components in the moduli space of theories with rank 2. We analyze in detail the limits of the M theory compactifications on a Klein bottle and on a Möbius strip which naively give type IIA string theory with an uncharged orientifold 8-plane carrying discrete RR flux. In order to consistently describe these limits we conjecture that this orientifold non-perturbatively splits into a D8-brane and an orientifold plane of charge (-1) which sits at infinite coupling. We construct the M(atrix) theory for M theory on a Klein bottle (and the theories related to it), which is given by a $2 + 1$ dimensional gauge theory with a varying gauge coupling compactified on a cylinder with specific boundary conditions. We also clarify the construction of the M(atrix) theory for backgrounds of rank 18, including the heterotic string on a circle.

Keywords: Superstring Vacua, M-Theory, String Duality, M(atrix) Theories.

Contents

1. Introduction

In this paper we discuss in detail the structure of the moduli space of nine dimensional $\mathcal{N} = 1$ supersymmetric backgrounds of M theory and string theory, and their M(atrix) theory construction. There are two main motivations for this study:

- The global structure of the moduli space of (maximally supersymmetric) toroidal compactifications of M/string theory has been studied extensively, and all of its corners have been mapped out. Less is known about compactifications which preserve only half of the supersymmetry (16 supercharges).[1] The moduli space of such nine dimensional backgrounds with rank 18 (including heterotic strings on a circle) has been extensively studied, and all its corners are understood; however it seems that no similar study has been done for backgrounds with lower rank (rank 10 or rank 2).

[1]Some of these backgrounds were studied in [1].

These backgrounds include the compactification of M theory on a Klein bottle and on a Möbius strip. In this paper we study the moduli space of these backgrounds in detail. We will encounter two surprises: we will see that the moduli space of backgrounds of rank 2 has two disconnected components,[2] and we will see that both for rank 2 and for rank 10 there is a region of the moduli space which has not previously been explored, and whose description requires interesting non-perturbative corrections to some orientifold planes in type IIA string theory.

- The theories we discuss (with rank n) have non-trivial duality groups of the form $\mathrm{SO}(n-1,1,\mathbb{Z})$. In their heterotic descriptions these are simply the T-duality groups. However, from the M theory point of view these dualities are quite non-trivial. In particular, it is interesting to study the manifestation of these duality groups in the M(atrix) theory description of these backgrounds; T-duality groups in space-time often map to interesting S-duality groups in the M(atrix) theory gauge theories. In this paper we will derive the M(atrix) theory for some of these backgrounds; the detailed discussion of the realization of the duality in M(atrix) theory is postponed to future work.

We begin in section 2 with a detailed discussion of the structure of the moduli space of 9d $\mathcal{N}=1$ backgrounds. We review the structure of the moduli space of rank 18 backgrounds, since this will have many similarities to the moduli spaces of reduced rank, and we then discuss in detail all the corners of the moduli spaces of reduced rank. In section 3 we construct the M(atrix) theory for the backgrounds corresponding to M theory on a cylinder (with a specific light-like Wilson line) and on a Klein bottle. The case of the cylinder has been constructed before [3], but we clarify its derivation and the mapping of parameters from space-time to the M(atrix) theory. Our construction for the Klein bottle is new. We end in section 4 with our conclusions and some open questions. Four appendices contain various technical details.

2. The moduli space of nine dimensional $\mathcal{N}=1$ backgrounds

In this section we review the moduli space of compactifications of string/M theory to nine dimensions which preserve $\mathcal{N}=1$ supersymmetry. On general grounds, if such a compactification has a rank n gauge group in its nine dimensional low-energy effective action, its moduli space takes the form $\mathrm{SO}(n-1,1,\mathbb{Z})\backslash \mathrm{SO}(n-1,1)/\mathrm{SO}(n-1)\times\mathbb{R}$, with the first component involving $n-1$ real scalars sitting in $n-1$ vector multiplets and the second component involving the scalar sitting (together with an additional U(1) vector field) in the graviton multiplet. In different regions of this moduli space there are different weakly coupled descriptions of the physics. We begin by reviewing the $n=18$ case which is well-known. We then discuss the cases of $n=2$ and $n=10$, where we will encounter some surprises and some regions which have not previously been analyzed.

[2] This was independently discovered by A. Keurentjes [2].

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2.1 A review of nine dimensional $\mathcal{N} = 1$ theories with rank 18

In this subsection we review the different regions of the moduli space of nine dimensional compactifications of M theory and string theory with 16 supercharges and a rank 18 group, including M theory on $\mathbb{R}^9 \times S^1 \times (S^1/\mathbb{Z}_2)$, the two heterotic strings compactified on a circle, the type I string on a circle and the type I' string. The results are mainly from [4] and [5]. The full moduli space for these theories is $\mathrm{SO}(17,1,\mathbb{Z})\backslash \mathrm{SO}(17,1)/\mathrm{SO}(17) \times \mathbb{R}$. Naturally, different descriptions are valid in different regions of this space. For simplicity, we shall restrict our discussion to the subspaces of moduli space that have enhanced $E_8 \times E_8$ and enhanced SO(32) gauge symmetries at low energies. These subspaces take the form $\mathrm{SO}(1,1,\mathbb{Z})\backslash \mathrm{SO}(1,1) \times \mathbb{R}$, so they are analogous to the rank 2 case, which is the main subject of this paper, and we will see many similarities between these two cases. However, a reader that is familiar with rank 18 compactifications is welcome to skip to the next subsection.

We define M theory on $\mathbb{R}^9 \times S^1 \times (S^1/\mathbb{Z}_2)$ by periodically identifying the coordinates x^9 and x^{10} with periodicities $2\pi R_9$ and $2\pi R_{10}$, and also identifying $x^{10} \simeq -x^{10}$ and requiring that the M theory 3-form $C_{\mu\nu\rho}$ change sign under this reflection. This space has two boundaries/orientifold planes, at $x^{10} = 0$ and at $x^{10} = \pi R_{10}$. The orientifold breaks half of the eleven dimensional supersymmetry, leaving 16 supercharges. This eleven dimensional orientifold is not anomaly-free: there is a gravitational anomaly in the 10d theory obtained by reducing on x^{10} that comes from the boundaries, and must be cancelled by additional massless modes that are restricted to the fixed planes. As in the 10 dimensional case, part of the anomaly can be cancelled by a generalized Green-Schwarz mechanism (using the $C_{\mu\nu(10)}$ form), and the remaining anomaly is cancelled by the addition of 496 vector multiplets. In ten dimensions this restricts the gauge group to be either SO(32) or $E_8 \times E_8$. However, in the 11d case the anomaly must be divided equally between the two fixed planes, so there is [4] an E_8 gauge group living at each orientifold plane. When the two radii are large (compared to the 11d Planck length), the low energy limit is described by 11d supergravity on the cylinder, coupled to two $\mathcal{N} = 1$ E_8 SYM theories on the two boundaries.

In the limit where R_9 is large and R_{10} is small, one obtains [4] the heterotic $E_8 \times E_8$ string, with string coupling $g_h = R_{10}^{3/2}$ and string length $R_{10}^{-1/2}$ (here and henceforth we suppress numerical constants of order one, and measure all lengths in 11d Planck units). This is a valid description of the physics as long as $R_{10} \ll 1$ and $R_9 \gg R_{10}^{-1/2}$.

When we continue to shrink R_{10} to make it smaller than R_9^{-2}, we reach a point where the x^9 circle becomes small compared to the string scale. At this point we must switch to the T-dual picture. Recall that the $E_8 \times E_8$ heterotic string with no Wilson lines is T-dual to itself, with an enhanced gauge symmetry at the self-dual radius. Thus, the appropriate description in this regime is once again the heterotic $E_8 \times E_8$ string, compactified on a circle of radius $R_{10}^{-1} R_9^{-1}$ with string coupling $g_{h'} = R_{10} R_9^{-1}$. This description is valid for $R_{10} \ll R_9$ and $R_9 \ll R_{10}^{-1/2}$. In the low-energy effective action, the $E_8 \times E_8 \times \mathrm{U}(1)^2$ gauge group is enhanced to $E_8 \times E_8 \times \mathrm{SU}(2) \times \mathrm{U}(1)$ along the line $R_9^2 R_{10} = 1$.

This description is valid for arbitrarily small R_{10}, so next we fix R_{10} and shrink R_9. This has the effect of increasing the string coupling in the T-dual heterotic picture. For

$R_9 \ll R_{10}$ we open up an extra dimension (as above) and get another region of moduli space that is also described by M theory on a cylinder. The length of the dual cylinder is $R_{10}^{1/2}R_9^{-1}$, the radius is $R_{10}^{-1}R_9^{-1}$ and the Planck length of this "other M theory" is $(l_p)_{M'} = R_9^{-1/3}R_{10}^{-1/6}$. Thus, this description is valid for $R_9 \ll R_{10} \ll R_9^{-4/5}$.

We are left with the region of $R_{10} \gg R_9^{-4/5}$ and $R_9 \ll 1$. This region is covered by the backgrounds that we get by reducing M theory on the periodic direction of the cylinder. This theory is known as type I' strings [6], and it may be viewed as an orientifold of type IIA string theory, obtained by dividing by worldsheet parity together with $\hat{x}^9 \to -\hat{x}^9$. The fixed points are now orientifold 8-planes ($O8$ planes) of type $O8^-$, which carry (-8) units of D8-brane charge. Tadpole cancellation then requires that this background must include also 16 D8-branes.

Another way to obtain the same type I' theory is as the T-dual of type I string theory. This is helpful in understanding an important feature of this background [5]. In type I strings, there are two diagrams that contribute to the dilaton tadpole — the disk and the projective plane — and these diagrams conspire to cancel, homogeneously throughout space-time. On the other hand, in type I' string theory there are two identical $O8$ planes at the boundaries, and 16 D8-branes that are free to move between the boundaries. As a result tadpole cancellation does not occur locally in this theory: the oriented disk diagram gets a contribution that is localized at the D8-branes, while the unoriented projective plane diagram gets contributions localized at the orientifold planes. Each D-brane cancels one-eighth of the contribution of an orientifold plane, and there is generally a gradient for the dilaton, whose exact form depends on the configuration of D8-branes.

The configuration of the branes between the orientifold planes also determines the low-energy gauge group of the background. One possible configuration of D-branes is to have 8 D8-branes on each $O8$ plane. In this case the dilaton is constant and there is an SO(16) gauge group at each of the boundaries. In all other cases the dilaton varies between the orientifold planes and the D-branes, and between the D-branes, with a gradient proportional to the inverse string length and to the local ten-form charge. This dilaton gradient causes the dilaton to diverge when the distance between two such planes is of order $g_{I'}^{-1}l_s$ (where $g_{I'}$ is the string coupling somewhere in the interval), imposing restrictions on the length of the interval and on the distances between the orientifold planes and the D-branes.

The last piece of information on type I' string theory that we need is that when the string coupling becomes infinite on one of the orientifold planes, there may be D0-branes that become massless there [7−9]. If there are n D8-branes on this orientifold plane, then these additional light degrees of freedom conspire to enhance the SO($2n$) gauge group to E_{n+1}. Specifically, in order to get an E_8 gauge group in this theory, we need to put 7 D8-branes on an orientifold plane, and one D8-brane away from it, precisely at a distance that will maintain the infinite string coupling at the $O8$ plane. If we do this at both ends (schematically: ($O8$+7D8)-D8-D8-($O8$+7D8)) we get an $E_8 \times E_8 \times$ U(1) $\times$ U(1) gauge group (in nine dimensions), providing the vacuum of type I' string theory that is dual to the heterotic $E_8 \times E_8$ string with no Wilson line turned on. The distance in string units between the two single D8-branes will be denoted by $x_{I'}$.

After this detour on the general properties of type I' string theory, let us now return to the compactification of M theory to this background. We can reach a type I' background in two ways: starting with the original M theory and reducing on x^9, or starting from the dual M theory and reducing on the periodic direction there. Both constructions give us a type I' theory in its $E_8 \times E_8$ vacuum, with couplings:

$$g_{I'1} = R_9^{3/2} \qquad (l_s)_{I'1} = R_9^{-1/2} \qquad R_{I'1} = R_{10} \tag{2.1}$$

$$g_{I'2} = R_9^{-1} R_{10}^{-5/4} \qquad (l_s)_{I'2} = R_{10}^{1/4} \qquad R_{I'2} = R_{10}^{1/2} R_9^{-1} \tag{2.2}$$

Note that we need to be careful about what we mean by $g_{I'}$, since the string coupling varies along the interval and diverges at the boundaries; what we will mean by $g_{I'}$ (here and in other cases with a varying dilaton) is the string coupling somewhere in the interior of the interval, and the differences between different points in the interval are of higher order in $g_{I'}$ so our expression is true in the weak $g_{I'}$ limit (which is the only limit where g is well-defined anyway). Naively, the first description is valid (except near the orientifold planes where the string coupling diverges) whenever $R_9 \ll 1$ and $R_{10} \gg R_9^{-1/2}$, and the second description is valid whenever $R_{10} \gg R_9^{-4/5}$ and $R_{10} \gg R_9^4$, but this would give an overlapping range of validity to the two different descriptions (which would also overlap with some of our previous descriptions). However, requiring that the distance between each D8 and the $O8$ should be such that the orientifold plane is at infinite coupling, we find $x_{I'}$ in the two cases to be

$$1) \qquad x_{I'1} = (R_{10} R_9^2 - 1) R_9^{-3/2} \tag{2.3}$$

$$2) \qquad x_{I'2} = (1 - R_9^2 R_{10}) R_9^{-1} R_{10}^{1/4} \tag{2.4}$$

Now, the regime of validity for each of the type I' backgrounds (defined by $(g_s)_{I'} \ll 1$ and $x_{I'} \geq 0$) is distinct; the first description is valid when $R_9 \ll 1$ and $R_{10} \leq R_9^{-2}$ and the second description is valid when $R_{10} \gg R_9^{-4/5}$ and $R_{10} \geq R_9^{-2}$. Furthermore, at the line $R_{10} R_9^2 = 1$, which was the line of enhanced symmetry in the heterotic $E_8 \times E_8$ string theory, we have $x_{I'} = 0$ for both backgrounds. On this line the two D8-branes coincide, enhancing the U(1) $\times$ U(1) symmetry to SU(2) $\times$ U(1) exactly as in the heterotic case. This implies that the two type I' backgrounds are actually related by an SU(2) gauge transformation, which is consistent with the fact that $x_{I'1}/(l_s)_{I'1} = -x_{I'2}/(l_s)_{I'2}$. We see that the line $R_{10} R_9^2 = 1$ continues to be a line of enhanced symmetry throughout the moduli space.

The backgrounds described thus far cover the whole moduli space of nine dimensional $\mathcal{N} = 1$ backgrounds with an $E_8 \times E_8$ gauge group. The top graph in figure 1 displays how these backgrounds fill in all possible values of the radii of the M theory we started with. Of course, the top part of this figure (above the dashed line) is related by an SU(2) gauge transformation to the bottom part, so the true moduli space is just half of this figure (above or below the dashed line).

Next, we turn to the subspace of the rank 18 backgrounds with an unbroken SO(32) gauge symmetry.[3] This subspace includes the type I' background in its SO(32) vacuum.

[3] Actually we will only describe one connected component of this subspace, which will turn out to be

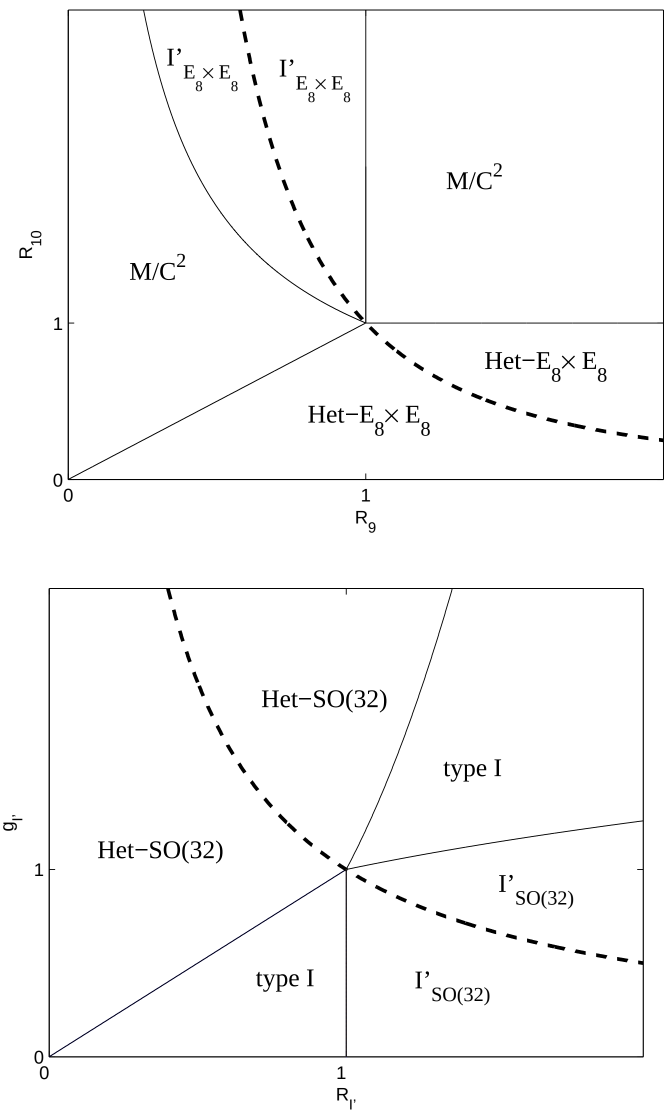

Figure 1: The $E_8 \times E_8$ and SO(32) subspaces of the moduli space of rank 18 compactifications of M theory and string theory. In the first graph the parameters are the period and length of the cylinder in Planck units as defined for the upper right compactification of M theory on a cylinder, and in the second they are the distance between the orientifold planes in units of the string length, and the string coupling on the $O8^-$ plane with the branes on it, of the type I' background appearing in the lower right corner. The dashed line in both graphs is the line of enhanced SU(2) symmetry; the regions of the graphs below and above this line are identified.

This vacuum is obtained by having all 16 D8-branes sit at one of the $O8^-$ planes (so that it has the same charges as an $O8^+$ plane). The dilaton then grows as we go from this $O8^-$ plane to the other one. We denote the distance between the orientifold planes as $R_{I'}$ (in

analogous to the rank 2 case. The other component has a discrete Wilson line that is only felt by spinor representations of SO(32).

string units) and the string coupling (defined, for instance, near the $O8^-$ plane with the branes) as $g_{I'}$. The regime of validity of this description is $1 \ll R_{I'} \ll g_{I'}^{-1}$.

For $R_{I'} \ll 1$, we need to switch to the T-dual of this description, which is the type I background with no Wilson lines. The string coupling in type I is $g_I = g_{I'}/R_{I'}$ and the radius of the compact circle is $R_I = R_{I'}^{-1}$. This description is thus valid for $g_{I'} \ll R_{I'} \ll 1$.

As we further decrease $R_{I'}$ we increase the coupling of the type I string theory, and eventually we should go over to the S-dual heterotic SO(32) string. The parameters of this heterotic background are $g_h = R_{I'}/g_{I'}$ and $R_h = 1/R_{I'}$, and its string length is $(l_s)_h = (g_{I'}/R_{I'})^{1/2}$, implying that it is a valid description for $R_{I'} \ll g_{I'} \ll R_{I'}^{-1}$.

On the line $g_{I'} = R_{I'}^{-1}$ we find $R_h = (l_s)_h$. This is a line of self T-duality and enhanced SU(2) gauge symmetry in the heterotic SO(32) string. We can now move to the T-dual picture, obtaining another heterotic SO(32) description. By going to the strong coupling limit of the new heterotic background, we get another type I background, and by another T-duality we get another regime described by the type I' background. These three types of backgrounds cover the whole SO(32) subspace of the moduli space, as presented in the second graph of figure 1.

In the heterotic SO(32) description we found that along the line $R_{I'} = g_{I'}^{-1}$ a U(1) symmetry gets enhanced to SU(2), by the usual mechanism of a winding mode becoming massless at the self-dual point. As in the previous example, the same enhancement occurs on this line also in other regions of the moduli space. In the type I' description, this is the line where the $O8$ plane without the branes is at infinite coupling. The half-D0-branes that live on the orientifold plane then become massless at this line, and they are responsible for the symmetry enhancement (from U(1) to $E_1 = \mathrm{SU}(2)$) in this description.

It is important to remember that the two branches depicted in figure 1 are just specific subspaces of the moduli space of rank 18 theories and that the rest can be reached by turning on Wilson lines in the heterotic or type I pictures, or equivalently moving around D8-branes in the type I' picture. Thus, the two plots in the figure are just two slices of the full moduli space. This is important to emphasize since in the next subsection a similar picture will arise, but in that case there are no Wilson lines to be turned on, so the two branches are actually two disconnected components of the 9d moduli space.

2.2 M theory on a Klein bottle and other rank 2 compactifications

We now wish to repeat the above analysis for M theory compactified on a Klein bottle (first considered in [10]) instead of a cylinder. One may start by asking if M theory makes sense at all on a Klein bottle. We will see that the answer is yes, and that this background arises as a strong coupling limit of consistent string theory backgrounds.

Let us begin by considering type IIB string theory gauged by the symmetry group $\mathbb{Z}_2 = \{1, H_9\Omega\}$, where Ω is worldsheet parity and H_9 is a shift in the 9th coordinate,

$$H_9\Omega \, : \begin{cases} X^\mu(z,\bar{z}) \simeq X^\mu(\bar{z},z), & \mu \neq 9 \\ X^9(z,\bar{z}) \simeq X^9(\bar{z},z) + 2\pi R_9. \end{cases} \tag{2.5}$$

This orientifold theory is derived by imposing $H_9\Omega = 1$ on the spectrum of the IIB theory compactified on a circle of radius $2R_9$; this breaks half of the supersymmetry (one of the gravitinos is projected out and the other one remains). In this case there are no fixed planes, and one does not need to add D-branes to cancel any tadpole, as explained in [10]. We refer to this background as the Dabholkar-Park background or the DP background.

We can compactify the DP background on an additional circle of radius $\tilde{R}_8$ in the x^8 direction, and T-dualize in this direction. Because of the orientation reversal in the original DP background, the resulting background is type IIA string theory gauged by

$$\begin{cases} X^\mu(z,\bar{z}) \simeq X^\mu(\bar{z},z), & \mu \neq 8,9 \\ X'^8(z,\bar{z}) \simeq -X'^8(\bar{z},z), & \\ X^9(z,\bar{z}) \simeq X^9(\bar{z},z) + 2\pi R_9. & \end{cases} \tag{2.6}$$

Thus, this describes a type IIA compactification on a Klein bottle ($K2$) of area $(2\pi R_8) \times (2\pi R_9)$, where $R_8 = l_s^2/\tilde{R}_8$, with an orientation reversal operation (see also [11]).[4] One can lift this theory to a background of M theory in the usual way by taking the strong coupling limit (the supersymmetry of this background prohibits a potential for the dilaton). The worldsheet parity is identified in M theory with flipping the sign of all components of the 3-form field. Thus, as promised, M theory on a Klein bottle naturally arises as a strong coupling limit of a string theory.

Consider now M theory compactified on a Klein bottle of radii $K2(R_{10}, R_9)$ measured in Planck units, namely with the identification

$$(X^{10}, X^9) \simeq (-X^{10}, X^9 + 2\pi R_9), \tag{2.7}$$

including a reversal of the 3-form field, and also with X^{10} periodically identified with radius R_{10}. If we shrink R_{10} we get a background that is described in detail in [12] and [13] (additional related analysis can be found in [14]). This background can be defined as type IIA string theory with a gauging of the symmetry

$$(-)^{F_L} \times (X^9 \to X^9 + 2\pi R_9), \tag{2.8}$$

where F_L is the space-time fermion number of the left-moving fields on the worldsheet. We refer to this background as the asymmetric orbifold of type IIA or AOA. The type IIA string coupling is $g_s \simeq R_{10}^{3/2}$, and $l_s \simeq R_{10}^{-1/2}$.

If we continue to shrink R_{10}, the circle becomes small (in string units) and we need to change to the T-dual description. It was noticed in [12] that the AOA background has an enhanced SU(2) symmetry point when $R_9 = l_s/\sqrt{2}$, and in [13] it was demonstrated that (as implied by this enhanced symmetry) this background is actually self-T-dual. In appendix C we explicitly evaluate the partition function of this model, showing that it is modular invariant and respects the T-duality. Therefore, when we continue to shrink R_{10} we should switch to a T-dual AOA description, which has $(R_9)_T \simeq 1/R_9 R_{10}$ and $(g_s)_T \simeq R_{10}/R_9$. This dual AOA description can in turn be lifted to a dual M theory on a Klein bottle. The regimes of validity of these different descriptions are exactly the same

[4] The orientation reversal leads to various periodicity conditions for the p-form fields, that are explained in detail in appendix B.

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(up to numerical constants of order one) as those we found in the previous subsection, with M theory on the Klein bottle replacing M theory on a cylinder, and the AOA background replacing the heterotic $E_8 \times E_8$ string.

Going back to M theory on a Klein bottle, what happens if we shrink R_9? Naively, we get a theory that is an orientifold of IIA on an $S^1/\mathbb{Z}_2$, with some boundary conditions at the fixed planes. However, we have a $\mathbb{Z}_2$ symmetry that exchanges the two boundaries, and since there cannot be any tadpole for the RR 10-form, we can only have neutral orientifold planes (with respect to the 10-form charge) at the boundaries, unlike the standard $O8^-$ planes (which carry (-8) units of charge) and $O8^+$ planes (which carry $+8$ units of charge).[5] For now we conjecture that the reduction in this direction leads to a background which we call the X-background, and that the line of enhanced symmetry $R_{10}R_9^2 = 1$ is also a line of enhanced symmetry in the X-background. Thus, the two remaining regions of the moduli space are covered by the X-background and its dual (obtained by reducing the dual M theory on the periodic cycle). In § 2.3 we shall propose a stringy description of this background.

The theories described above (starting with M theory on a Klein bottle) cover a complete moduli space of the form $\mathrm{SO}(1,1,\mathbb{Z})\backslash\mathrm{SO}(1,1) \times \mathbb{R}$, as shown on the top graph of figure 2. However, there are several additional string backgrounds with 16 supercharges and a rank 2 gauge symmetry that were not included so far, so they must be in a separate component of the moduli space. The first is the DP background described above. Next, there is the orientifold of type IIA on an interval $S^1/\mathbb{Z}_2$, with an $O8^-$ plane on one end and an $O8^+$ plane on the other [15]. We refer to this background as the $O8^\pm$ background, and denote the length of the interval in string units by $\pi R_\pm$. No D-branes are required for tadpole cancellation here. However, the dilaton does have a gradient (identical to that in the SO(32) background of the type I' string) that puts a limit on the maximum length of the interval, of the form $R_\pm < 1/g_\pm$ (where $g_\pm$ is again defined as the string coupling somewhere in the interval). This background was studied in [15], where it was shown to be T-dual to the DP background. An important feature of this background is that when the distance between the orientifold planes is such that the coupling on the $O8^-$ is infinite, there are half-D0-branes stuck on the orientifold that become massless. Locally, this is exactly the same as in the type I' SO(32) vacuum; in both cases the fractional branes enhance the $\mathrm{U}(1)^2$ symmetry to $\mathrm{SU}(2) \times \mathrm{U}(1)$.

Finally, one other background can be obtained by gauging type IIB string theory by the symmetry (2.8). We call this background the Asymmetric Orbifold of type IIB or AOB. Recalling the following property of IIB strings

$$(-1)^{F_L} = S\Omega S \qquad (2.9)$$

(where S is the S-duality transformation of type IIB string theory) and applying the adiabatic argument of [14], we see that this background is S-dual to the DP background. Exactly like the AOA background, the AOB background is self-T-dual [13] and has an enhanced SU(2) symmetry when the radius of the circle is $l_s/\sqrt{2}$.

[5] Notice that this contradicts a suggestion occasionally found in the literature, that in this limit we get a background defined by IIA on an $S^1/\mathbb{Z}_2$ with an $O8^-$ plane on one end and an $O8^+$ plane on the other.

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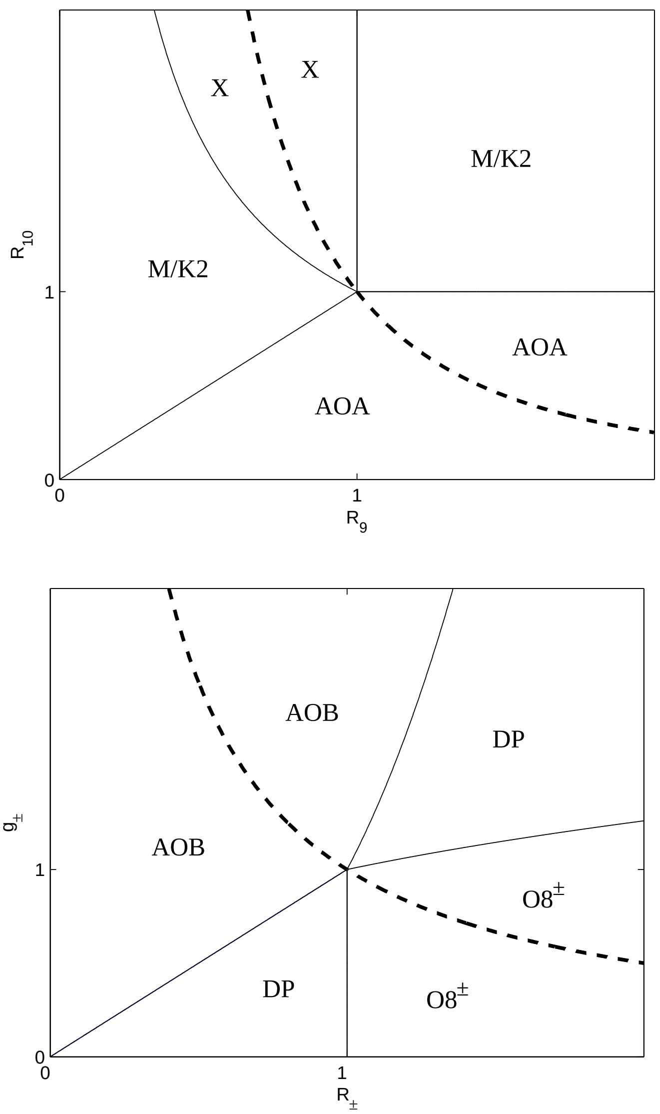

Figure 2: The two disconnected components of the moduli space including the backgrounds of subsection 2.2. In the first graph the parameters are the periods of the Klein bottle in Planck units as defined in (2.7) for the upper right M theory description. In the second graph, the parameters are the radius of the circle in string units and the string coupling (in the middle of the interval) for the $O8^\pm$ background on the lower right corner. The dashed line in both graphs is the line of enhanced SU(2) gauge symmetry, and the backgrounds below the line are identified with the backgrounds above the line.

We can now describe the second component of the moduli space. Start with the $O8^\pm$ background with string coupling $g_\pm$ and radius $R_\pm$ (measured in string units). This is a valid description for $g_\pm \ll 1/R_\pm$ and $R_\pm \gg 1$. As we decrease $R_\pm$, we need to T-dualize, as in [15], and we get the DP background with radius[6] $1/R_\pm$ and string coupling $g_\pm/R_\pm$.

[6]We continue to ignore numerical constants of order one.

Figure 3: In the X region, the Klein bottle looks like a long tube between two cross-caps.

The DP description is valid as long as $g_\pm \ll R_\pm \ll 1$. We can further decrease $R_\pm$ to the point where the DP string coupling is too large. At this point we turn to the S-dual picture, which is the asymmetric orbifold of type IIB (AOB). The string coupling is now $R_\pm/g_\pm$, and the S-dual string length is $(g_\pm/R_\pm)^{1/2}$, such that the radius $1/R_\pm$ in string units is now $(g_\pm R_\pm)^{-1/2}$. This is a good description for $g_\pm \ll 1/R_\pm$ and $R_\pm \ll g_\pm$.

We continue next by increasing $g_\pm$ to the point where the AOB radius becomes small. Then we must use the self T-duality of this background to arrive at a dual AOB description, with radius $(g_\pm R_\pm)^{1/2}$ (in units of its string length) and string coupling $g_\pm^{-1/2} R_\pm^{3/2}$. This description is good for $g_\pm \gg 1/R_\pm$ and $g_\pm \gg R_\pm^3$. The next step is to take the S-dual of the dual AOB background. This gives another DP background with radius (in units of its string length) $g_\pm^{3/4} R_\pm^{-1/4}$ and coupling $g_\pm^{1/2} R_\pm^{-3/2}$. This description is valid for $g_\pm \gg R_\pm^{1/3}$ and $g_\pm \ll R_\pm^3$. The last step is to T-dualize the new DP background to a dual $O8^\pm$ background, with radius $g_\pm^{-3/4} R_\pm^{1/4}$ (in units of its string length) and coupling $g_\pm^{-1/4} R_\pm^{-5/4}$. This description is valid for $g_\pm \ll R_\pm^{1/3}$ and $g_\pm \gg 1/R_\pm$.

These six string theories cover the entire range of values of $(g_\pm, R_\pm)$. The reader should notice that the line of enhanced SU(2) symmetry in the $O8^\pm$ background, $R_\pm \sim 1/g_\pm$, smoothly goes into the line of enhanced SU(2) symmetry of the AOB background [12], as in the previous cases we discussed.

Figure 2 summarizes the structure of the moduli space. As promised, it has two disconnected components in the nine dimensional sense. Note that the nine dimensional low-energy effective action on the two components is identical, but the massive spectrum is different. The components can be related by compactifying on an additional circle and performing a T-duality, but they are not connected as nine dimensional backgrounds. There is an obvious relation between each of these components and one of the subspaces of the moduli space of rank 18 compactifications discussed in the previous subsection and depicted in figure 1.

2.3 The X background demystified

In the previous subsection we left open the description of the limit of the Klein bottle where R_9 is small and R_{10} is large. In this limit the Klein bottle geometrically looks as in figure 3 (though the geometric description is not really valid at distances smaller than the 11d Planck scale). This limit should correspond to some 10 dimensional string theory; let us collect some features of this theory:

- After reducing M theory on the small circle R_9, we expect to obtain a type IIA string theory that lives on $\mathbb{R}^{8,1} \times I$ where I is an interval. On each boundary of the interval we should have an orientifold plane; however these orientifold planes are non-standard because the orientifolding is accompanied by a half-shift on the M theory circle. This half-shift modifies, among other things, the properties of D0-branes near the orientifold. One can think of this shift as a discrete RR flux characterizing the orientifold plane, which changes its charge from the usual charge of (-8) to zero [16]. We will denote such orientifold planes by $O8^0$.

- The fact that the orientifold carries no D8-brane charge is consistent with tadpole cancellation of the 10-form field and the dilaton. Note that there is a $\mathbb{Z}_2$ symmetry exchanging the two ends of the interval, so the two orientifold planes must carry the same charge (unlike the case in the $O8^{\pm}$ background described in the previous subsection).

- Naively one expects that in the limit of small R_9 the 10 dimensional string theory can be made very weakly coupled, and the description should involve the usual type IIA string theory, at least far from the boundaries of the interval (where there may or may not be large quantum corrections). We will see that there are some regions of the moduli space where this naive expectation fails.

- The discussion of the previous subsection suggests that we should have an enhanced SU(2) gauge symmetry when the interval is of size $R \sim 1/g_s$ (in type IIA string units). One may expect this enhanced symmetry to come from half-D0-branes on the orientifold planes, as in some of the examples discussed in the previous subsections. However, since we have a $\mathbb{Z}_2$ symmetry relating the two orientifolds, it is hard to imagine how the $\mathrm{U}(1)^2$ group would be enhanced to $\mathrm{U}(1) \times \mathrm{SU}(2)$ rather than to the more symmetric $\mathrm{SU}(2)^2$.

The last item above suggests some modification of the naive picture of this background. The picture that we will suggest for the correct description of this background is based on two facts:

- In our analogy between the rank 18 and the rank 2 theories, the X background plays the same role as the type I' $E_8 \times E_8$ background. In this background the two orientifold planes are always at infinite coupling, and the enhanced SU(2) symmetry arises when two D8-branes in the middle of the interval come together [9].

- In some cases, when a standard ($O8^-$) orientifold plane of charge (-8) is at infinite coupling, it can emit a D8-brane into the bulk, leaving behind an orientifold plane of charge (-9) which has no perturbative description (and which always sits at infinite coupling). This phenomenon cannot be seen in perturbation theory, but it can be deduced from an analysis of D4-brane probes [7].

Our suggestion is that each $O8^0$ plane non-perturbatively emits a D8-brane and becomes an $O8^{(-1)}$ plane which always sits at infinite coupling. The moduli space and 10-form fluxes of this system are then identical to those of the $E_8 \times E_8$ type I' background, with the $O8^{(-1)}$ plane playing the same role as the $O8^-$ plane with 7 D8-branes on it. Now, the gauge symmetry is enhanced to SU(2) in a $\mathbb{Z}_2$-symmetric manner, when the D8-branes meet in the middle of the interval, and this enhancement is perturbative (it can happen at weak coupling). Denoting the string coupling in the region between the two emitted D8-branes by g_s (it does not vary in the interval between the D8-branes), the distance of each brane from the respective $O8^{(-1)}$ plane is $\sim 1/g_s$. Thus, when the branes meet, our interval indeed has a size proportional to $1/g_s$ as required.

We suggest that this is the correct description of $O8^0$ orientifold planes. Note that such a large non-perturbative correction to the description of high-dimensional orientifold planes is not surprising; already in the case of $O7$ planes it is known [17] that they non-perturbatively split into two 7-branes, and the corrections to $O8$ planes are expected to be even larger. Our suggestion implies non-trivial corrections to compactifications of M theory involving crosscaps (similar corrections in M theory should also occur in a compactification on a Möbius strip, as will be discussed in the next subsection). These corrections are similar to the ones that occur for M theory on a cylinder with no Wilson lines. Note that when we are close to the enhanced SU(2) point, the corrections to the naive M theory picture shown in figure 3 are not just localized near the cross-caps as one may naively expect, but the string coupling actually varies along the whole interval.

In order for this proposal to be consistent, there should be no massless fractional D0-branes on the $O8^{(-1)}$ planes, which would lead to more enhanced symmetries than we need. Because of the shift in the M theory circle involved in the orientifolding, the radius of the cross-cap is actually half of the radius of the M theory circle in the bulk, which implies that only even momentum modes (in units of the minimal momentum on a standard orientifold plane) are allowed there. Hence, there are no half D0-branes in backgrounds with $O8^0$ orientifolds (or $O8^{(-1)}$ orientifolds).

2.4 M theory on a Möbius strip and other rank 10 compactifications

There is one additional disconnected component of the moduli space of nine dimensional compactifications with $\mathcal{N} = 1$ supersymmetry. Consider the heterotic $E_8 \times E_8$ string compactified to nine dimensions on a circle of radius R_9. One can consider an orbifold of this theory generated by switching the two gauge groups together with a half-period-shift of x^9. This theory is known as the CHL string; as usual one should add twisted sectors for the consistency of the orbifold (for more details see [18, 19]). This leads to a nine dimensional compactification with a rank 10 gauge group. We will focus on the subspace of the moduli space of this theory in which the E_8 symmetry is unbroken.

An orbifold of the $E_8 \times E_8$ heterotic string on a circle can also be viewed as an orbifold of M theory compactified on a cylinder. Begin with M theory compactified on the cylinder

$$x^9 \simeq x^9 + 2\pi R_9 \quad , \quad x^{10} \in \left[-\frac{\pi}{2}R_{10}, \frac{\pi}{2}R_{10}\right]. \tag{2.10}$$

The action on the heterotic string which we described above lifts in M theory to

$$x^9 \simeq x^9 + \pi R_9 \quad , \quad x^{10} \simeq -x^{10}. \tag{2.11}$$

Figure 4: M theory on a Möbius strip describes the strong coupling limit of the CHL string.

Upon identifying points related by this transformation we obtain a Möbius strip, with a cross-cap at $x^{10} = 0$ and a boundary at $x^{10} = \pi R_{10}/2$, as depicted in figure 4. Thus, this component of the moduli space is generated by various limits of M theory compactified on the Möbius strip [10, 20]. Notice that anomaly cancellation as in [4] tells us that the single boundary of the Möbius strip carries an E_8 gauge symmetry, which is consistent with the low-energy gauge symmetry of the CHL string.

So far we have good descriptions of the regions of moduli space where both R_9 and R_{10} are large, and when R_{10} is small (leading to the CHL string). It is natural to ask what string theory backgrounds are obtained when we reduce on the other direction, $R_{10} \gg l_P \gg R_9$. In this limit, the cross-cap and the boundary (which is topologically $\mathbb{R}^{8,1} \times S^1$) of figure 4 are very far from each other. The boundary becomes a usual $O8^-$ plane of IIA string theory. Our previous description of E_8 symmetries in type IIA string theory implies that this $O8^-$ plane has 7 D8-branes on top of it, and there is an additional one displaced such that the system $O8^-$+7D8 is at infinite string coupling. The cross-cap should behave exactly as in the case of the Klein bottle, described in the previous subsection. Therefore, the naive $O8^0$ plane emits an extra D8-brane and becomes an $O8^{(-1)}$ plane at infinite string coupling.

When we approach the point $R_{10}R_9^2 \sim 1$, we find that the two D8-branes in the bulk come together, and enhance the $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry to $\mathrm{SU}(2)\times\mathrm{U}(1)$. As in our previous examples, the same symmetry enhancement arises also for small R_{10}, where it arises at the self-dual radius of the perturbative CHL string (for an exhaustive analysis of the momentum lattices in toroidal compactifications of CHL strings see [21]). In fact, it is just the same as the SU(2) enhanced symmetry of the $E_8 \times E_8$ string at the self dual radius (the gauge bosons are BPS and survive the CHL projection). This is another consistency check on our proposal for the behavior of the cross-cap in M theory.

We conclude that the moduli space of M theory on a Möbius strip has a line of enhanced SU(2) symmetry, and that all of its limits may be understood (with some strongly coupled physics occurring in the IIA limit). This moduli space is depicted in figure 5, where the type IIA orientifold limit is denoted by $(1/2)X$. In fact, this moduli space is identical in structure to the other moduli spaces we encountered in our survey, arising from M theory on a cylinder and on a Klein bottle.

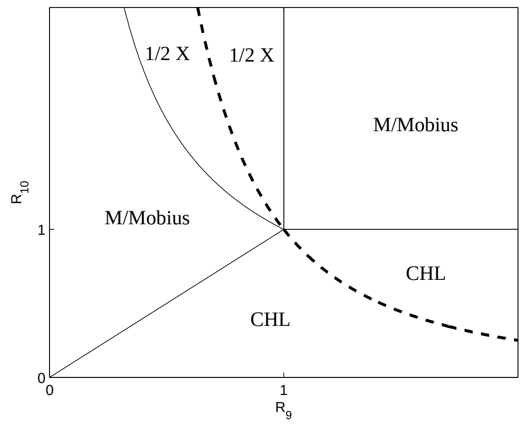

Figure 5: The moduli space of compactifications of M theory on a Möbius strip. The parameters are the period and length of the strip in Planck units as defined for the upper right M theory. The dashed line is the line of enhanced SU(2) symmetry.

3. The M(atrix) theory description of M theory on a cylinder and on a Klein bottle

In this section we describe the M(atrix) theory [22] which is the discrete light-cone quantization (DLCQ) of some of the theories described in the previous section – M theory compactified on a cylinder and on a Klein bottle. We begin by considering the case of a cylinder [3], and then move on to the Klein bottle. We review in detail the case of the cylinder because of the great similarity between these two compactifications (as described in the previous section), which is useful in the construction of the correct M(atrix) theory of the Klein bottle.

3.1 The M(atrix) theory of M theory on a cylinder

This subsection is based on [3], with the addition of a systematic derivation of their M(atrix) theory and of some small corrections to the identifications of parameters presented in that paper.

M(atrix) theory is the discrete light-cone quantization of M theory backgrounds [23]; it provides the Hamiltonian for these theories compactified on a light-like circle, with N units of momentum around the circle. In general, such a DLCQ description is very complicated. However, in some M theory backgrounds it simplifies, because a light-like circle may be viewed [24, 25] as a limit of a very small space-like circle, and M theory on a very small space-like circle is often very weakly coupled. This leads to a simple description of the DLCQ Hamiltonian, in which most of the degrees of freedom of M theory decouple. In particular, this is the case for the M(atrix) theory of M theory itself, which is given just by a maximally supersymmetric U(N) quantum mechanical gauge theory, and for the M(atrix) theory of M theory compactified on a two-torus, which is given by the maximally supersymmetric U(N) $2+1$ dimensional gauge theory, compactified on a dual torus.

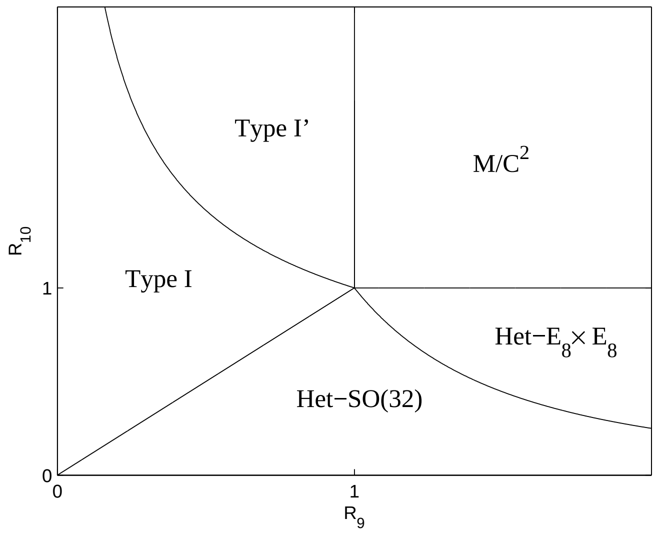

Figure 6: The subspace of the moduli space of M theory on a cylinder, where all backgrounds include a Wilson line breaking the gauge group to $\mathrm{SO}(16) \times \mathrm{SO}(16)$. The parameters in the plot are the period R_9 and length R_{10} of the cylinder in Planck units.

The generic simplicity of M(atrix) theory is based on the fact that M theory compactified on a small space-like circle becomes a weakly coupled type IIA string theory. However, when boundaries are present in the M theory compactification, they usually destroy this simplicity. For instance, as is evident from figure 1, if we take M theory on $S^1/\mathbb{Z}_2$ and compactify it further on a very small space-like circle, we do not obtain a weakly coupled background (but, rather, we obtain M theory on a dual cylinder). Thus, generically the DLCQ of M theory backgrounds with boundaries is very complicated. However, there is an extra degree of freedom one can use in the DLCQ constructions, which is a Wilson line along the light-like circle; such a Wilson line becomes irrelevant in the large N limit of M(atrix) theory in which it provides a light-cone quantization of the original background (without the light-like circle), but it can have large effects for finite values of N. In the case of M theory on $S^1/\mathbb{Z}_2$, as we discussed above, the theory compactified on an additional small circle is generally strongly coupled (see [26] for a recent discussion), except when we have a Wilson line breaking the $E_8 \times E_8$ symmetry to $\mathrm{SO}(16) \times \mathrm{SO}(16)$. The moduli space of this subspace of compactifications on a cylinder is drawn in figure 6; as can be seen in this figure, the limit of a small space-like circle leads in this case to a weakly coupled type I string theory (with a Wilson line breaking SO(32) to $\mathrm{SO}(16) \times \mathrm{SO}(16)$). The M(atrix) theory for M theory on an interval with this specific light-like Wilson line is then again a simple theory [27–31] — the decoupled theory of N D1-branes in this type I background, which is simply an $\mathrm{SO}(N)$ $1+1$ dimensional $\mathcal{N} = (0, 8)$ supersymmetric gauge theory on a circle, coupled to 32 real left-moving fermions in the fundamental representation (coming from the D1-D9 strings), half of which are periodic and half of which are anti-periodic [5].

Now we can move on to the case we are interested in, which is the M(atrix) theory of M theory on $\mathbb{R}^9 \times S^1 \times (S^1/\mathbb{Z}_2)$, with an arbitrary Wilson line W for the $E_8 \times E_8$ gauge group on the S^1. To construct the M(atrix) theory we should again consider the limit of this theory on a very small space-like circle [24, 25], with a particular scaling of the size of the cylinder as the size of this extra circle goes to zero. Again, in general this limit gives a strongly coupled theory, except in the case where we have an additional Wilson line breaking the $E_8 \times E_8$ gauge group to $\mathrm{SO}(16) \times \mathrm{SO}(16)$ on the additional circle (the original Wilson line W must commute with this Wilson line in order to obtain a weakly coupled description).[7] In such a case we obtain precisely the theory described in the previous paragraph, compactified on an additional very small circle with a Wilson line $\tilde{W}$ (which is the translation of the original Wilson line from the $E_8 \times E_8$ variables of the original M theory to the SO(32) variables of the dual type I background). Since the additional circle is very small we need to perform a T-duality on this circle. We then obtain a type I' theory of the type described in the previous section, still compactified on a circle with the $\mathrm{SO}(16) \times \mathrm{SO}(16)$ Wilson line, and with the positions of the D8-branes determined by the eigenvalues of the Wilson line $\tilde{W}$. The D1-branes we had before now become D2-branes which are stretched both along the interval between the orientifold planes and along the additional circle.

The usual derivation of M(atrix) theory [24, 25] shows that the M(atrix) theory is precisely the decoupled theory living on these D2-branes, in the limit that the string mass scale goes to infinity keeping the Yang-Mills coupling constant on the D2-branes, which is proportional to

$$g_{\mathrm{YM}}^2 \propto g_s/l_s, \tag{3.1}$$

fixed. The D2-brane lives on a cylinder, with a circle of radius R_1 (related to the parameters of the original M theory background by $R_1 = l_p^3/R_{10}R$, where $1/R$ is the energy scale associated with the light-like circle) and an interval of length πR_2 (given by $R_2 = l_p^3/R_9 R$). In the standard case of toroidal compactifications, only the disk contributions to the D2-brane action survive in this limit, giving a standard supersymmetric Yang-Mills theory, with Yang-Mills coupling $g_{\mathrm{YM}}^2 = R/R_9 R_{10}$. However, in our case it turns out that some contributions to the D2-brane action from Möbius strip diagrams also survive; this is evident from the fact that g_s in the type I' background is generally not a constant, leading through (3.1) to a non-constant Yang-Mills coupling. This was taken into account in [3], where the Lagrangian for any distribution of D8-branes was obtained, and it was shown that the Möbius strip contributions are crucial to cancel anomalies in the gauge theory.

There is one special case when the Möbius contributions are absent; this is the case when the type I' background has a constant dilaton, with eight D8-branes on each orientifold plane. According to the discussion above, this case provides the DLCQ description of M theory on a cylinder with a Wilson line breaking the gauge symmetry to $\mathrm{SO}(16) \times \mathrm{SO}(16)$ (so that we are at some point on the moduli space of figure 6), and with an additional light-like Wilson line which breaks the gauge symmetry in the same way. We begin by describing

[7] In principle we could also have a light-like Wilson line for the other two U(1) gauge fields appearing in the low-energy nine dimensional effective action, but we will not discuss this here.

this special case. In this case the theory on the D2-branes away from the orientifold planes is just the standard maximally supersymmetric $2+1$ dimensional $\mathrm{U}(N)$ Yang-Mills theory, with a gauge coupling related to the parameters of the original M theory background by $g^2_{\mathrm{YM}} = R/R_9R_{10}$ (which is the same relation as in toroidal compactifications). The field content of this theory includes a gauge field A_μ, seven scalar fields X^j and eight Majorana fermions ψ_A. The boundary conditions project the $\mathrm{U}(N)$ gauge group to $\mathrm{SO}(N)$. In addition, the D2-D8 strings give rise to 8 complex chiral fermions in the fundamental representation χ_k ($k=1,\cdots,8$) at the boundary $x^2=0$ and 8 additional fermions $\tilde{\chi}_k$ ($k=1,\cdots,8$) at the other boundary $x^2=\pi R_2$. The action is given by

$$S = \int dt \int_0^{2\pi R_1} dx^1 \Bigg[\int_0^{\pi R_2} dx^2 \frac{1}{2g^2_{\mathrm{YM}}} \mathrm{Tr} \Bigg(-\frac{1}{2} F_{\mu\nu}F^{\mu\nu} + (D_\mu X^j)^2 + \\ +\frac{1}{2}[X^j,X^i][X^j,X^i] + i\bar{\psi}_A\gamma^\alpha D_\alpha \psi_A - i\bar{\psi}_A \gamma^i_{AB}[X_i,\psi_B] \Bigg) + \\ +i\sum_{k=1}^{8} \bar{\chi}_k(\partial_- + iA_-|_{x^2=0})\chi_k + i\sum_{k=1}^{8} \bar{\tilde{\chi}}_k(\partial_- + iA_-|_{x^2=\pi R_2})\tilde{\chi}_k \Bigg], \tag{3.2}$$

where D_μ is the covariant derivative for the adjoint representation, $\partial_- \equiv \partial_t - \partial_1$ and similarly for A_-. Our conventions for fermions and spinor algebra are summarized in appendix A. Due to the light-like Wilson line described above, the fermions χ_k are periodic around the x^1 circle, while the fermions $\tilde{\chi}_k$ are anti-periodic; this can alternatively be described by adding a term $\frac{1}{4\pi R_1}$ to the kinetic term of the $\tilde{\chi}_k$ in (3.2).

The boundary conditions can be determined by consistency conditions for D2-branes ending on an $O8^-$ plane. For the bosonic fields, at both boundaries, the boundary conditions take the form

$$\begin{aligned} X^j &= (X^j)^T, & \partial_2 X^j &= -(\partial_2 X^j)^T, \\ A^{0,1} &= -(A^{0,1})^T, & \partial_2 A^{0,1} &= (\partial_2 A^{0,1})^T, \\ A^2 &= (A^2)^T, & \partial_2 A^2 &= -(\partial_2 A^2)^T. \end{aligned} \tag{3.3}$$

These boundary conditions break the $\mathrm{U}(N)$ bulk gauge group to $\mathrm{SO}(N)$. The zero modes along the interval are $\mathrm{SO}(N)$ gauge fields $A^{0,1}$ and eight scalars in the symmetric representation of $\mathrm{SO}(N)$ coming from A^2 and X^j. For the fermions, the boundary conditions take the form

$$\psi_A = -i\gamma^2\psi_A^T, \qquad \partial_2\psi_A = i\gamma^2\partial_2\psi_A^T. \tag{3.4}$$

The zero modes for the right-moving fermions are in the adjoint representation of $\mathrm{SO}(N)$, and those of the left-moving fermions are in the symmetric representation. This leads to an anomaly in the low-energy $1+1$ dimensional $\mathrm{SO}(N)$ gauge theory, which is precisely cancelled by the 16 additional chiral fermions in the fundamental representation; this cancellation occurs locally at each boundary.

The bulk theory has eight $2+1$ dimensional supersymmetries (16 real supercharges), but the boundary conditions and the existence of the D2-D8 fermions break this to a

$\mathcal{N} = (0,8)$ chiral supersymmetry (SUSY) in $1+1$ dimensions. The supersymmetry transformation rules are given by

$$\begin{aligned}
\delta_\epsilon A_\alpha &= \frac{i}{2}\bar{\epsilon}_A \gamma_\alpha \psi_A, \\
\delta_\epsilon X^i &= -\frac{1}{2}\bar{\epsilon}_A \gamma^i_{AB} \psi_B, \\
\delta_\epsilon \psi_A &= -\frac{1}{4} F_{\alpha\beta}\gamma^{\alpha\beta}\epsilon_A - \frac{i}{2} D_\alpha X_i \gamma^\alpha \gamma^i_{AB}\epsilon_B - \frac{i}{4}[X_i, X_j]\gamma^{ij}_{AB}\epsilon_B, \\
\delta_\epsilon \chi_k &= 0\,, \\
\delta_\epsilon \tilde{\chi}_k &= 0\,.
\end{aligned} \tag{3.5}$$

These transformation rules are consistent with the boundary conditions (3.3), (3.4) only for

$$\epsilon_A = i\gamma^2 \epsilon_A\,, \tag{3.6}$$

and thus indeed the boundary conditions preserve only 8 of the original 16 supercharges. Decomposing the fields by their γ^2 eigenvalues $\pm i$:

$$\epsilon_A = \begin{pmatrix} \epsilon^+_A \\ \epsilon^-_A \end{pmatrix}, \qquad\qquad \psi_A = \begin{pmatrix} \psi^+_A \\ \psi^-_A \end{pmatrix}, \tag{3.7}$$

it follows that the unbroken SUSY is for ϵ^-_A.

In the more general case, as described above, we consider a similar background but with the D8-branes at arbitrary positions in the bulk. One obvious change is then that the chiral fermions χ and $\tilde{\chi}$ are no longer localized at the boundaries but rather at the positions of the D8-branes. More significant changes are that the varying dilaton leads to a varying gauge coupling constant, and the background 10-form field in the type I' background leads to a Chern-Simons term, which is piece-wise constant along the interval. The most general Lagrangian was written in [3], where it was also verified that it is supersymmetric and anomaly-free. The relation between the positions of the D8-branes and the varying coupling and 10-form field, which in the bulk string theory comes from the equations of motion, is reproduced in the gauge theory by requiring that there is no anomaly in arbitrary $2+1$ dimensional gauge transformations.

For the purposes of comparison with the Klein bottle case that we will discuss in the next subsection, it is useful to consider the special case where there is no Wilson loop on the cylinder. This gives, in particular, the M(atrix) theory of the $E_8 \times E_8$ heterotic string compactified on a circle with no Wilson line. In this case we have a configuration where all D8-branes are on the same orientifold plane, say the one at $x^2 = \pi R_2$. The action in this special case may be written in the form (now denoting the scalar fields by Y^i and the adjoint fermions by Ψ_A)

$$
\begin{aligned}
S = \int dt \int_0^{2\pi R_1} dx^1 \Bigg[\int_0^{\pi R_2} dx^2 \frac{1}{4g_{\rm YM}^2} \operatorname{Tr}\Bigg(& -z(x^2) F_{\mu\nu}F^{\mu\nu} + 2z^{1/3}(x^2)(D_\mu Y^j)^2 + \\
& + z^{-1/3}(x^2)[Y^j, Y^i][Y^j, Y^i] + 2iz^{1/3}(x^2)\bar{\Psi}_A \gamma^\alpha D_\alpha \Psi_A + \frac{dz^{1/3}(x^2)}{dx^2}\bar{\Psi}_A \Psi_A \\
& - 2i\bar{\Psi}_A \gamma^i_{AB}[Y_i, \Psi_B] + \frac{4}{3}\frac{dz(x^2)}{dx^2}\epsilon^{\alpha\beta\gamma}\left(A_\alpha \partial_\beta A_\gamma + i\frac{2}{3}A_\alpha A_\beta A_\gamma\right)\Bigg) \\
& + i\sum_{k=1}^{8} \bar{\chi}_k(\partial_- + iA_-|_{x^2=\pi R_2})\chi_k + i\sum_{k=1}^{8} \bar{\tilde{\chi}}_k(\partial_- + iA_-|_{x^2=\pi R_2})\tilde{\chi}_k \Bigg].
\end{aligned} \tag{3.8}
$$

The varying coupling constant is given by

$$
z(x^2) = 1 + \frac{6g_{\rm YM}^2}{\pi}\left(x^2 - \frac{\pi R_2}{2}\right), \tag{3.9}
$$

and the coupling grows as we approach the $O8^-$ plane with no D8-branes on it. Here we arbitrarily defined $g_{\rm YM}$ to be the effective coupling constant at the middle of the interval (other choices would modify the constant term in (3.9)). The linear term in (3.9) is related to the background 10-form field. The effective Yang-Mills coupling constant is

$$
\begin{aligned}
(g_{\rm YM}^{\rm eff})^2 &= \frac{g_{\rm YM}^2}{1 + 6g_{\rm YM}^2(x^2 - \pi R_2/2)/\pi} \\
&= \frac{1}{1/g_{\rm YM}^2 + 6(x^2/\pi - R_2/2)}.
\end{aligned} \tag{3.10}
$$

This description is valid as long as the Yang-Mills coupling constant does not diverge anywhere, namely for $g_{\rm YM}^2 < 1/(3R_2)$. This is the same condition as the string coupling not diverging in the type I' string theory which we used for deriving this action. The boundary conditions on the fields are essentially the same as before, but there as some modifications in the boundary conditions which involve derivatives and in the SUSY transformation laws. The same modifications will appear in the Klein bottle case that we will discuss in the next subsection, and we will discuss them explicitly there. The fermions χ_k are still periodic and the fermions $\tilde{\chi}_k$ anti-periodic due to the light-like Wilson line.

Note that in the two special cases that we described, (3.2) and (3.8), the theory is exactly free for $N = 1$ (as was the original BFSS M(atrix) theory), since the gauge fields $A_{0,1}$ vanish at both boundaries; however, this is not the case at more general points on the moduli space. The usual argument that $N = 1$ DLCQ theories should be free is that $N = 1$ is the minimal amount of possible light-like momentum, so the theory must contain a single particle with this momentum and no interactions. However, this is no longer true in the presence of generic light-like Wilson lines, which modify the quantization of the light-like momentum for charged states.

3.2 The M(atrix) theory of the Klein bottle compactification

In this subsection we describe the M(atrix) theory of M theory on a Klein bottle, and we will see that it is very similar to the case described in the previous subsection.[8] Again, to derive the M(atrix) theory we need to consider M theory on a Klein bottle times a very small space-like circle. Now we do not need to add any Wilson lines to get a weakly coupled theory; instead we directly obtain N D0-branes in the weakly coupled type IIA string theory on a Klein bottle that was mentioned in the previous section, in the limit in which the Klein bottle has a very small size. We then need to perform two T-dualities to go back to a finite-size compact manifold. The relevant T-dualities were already described in section 2.2: one T-duality (which is straightforward) leads to the DP background, and the next leads to the $O8^{\pm}$ background. Thus, the M(atrix) theory is the decoupled theory on $2N$ D2-branes stretched between an $O8^-$ plane and an $O8^+$ plane[9] (we obtain $2N$ D2-branes due to the presence of D0-branes as well as their images on the original Klein bottle). This theory is very similar to the theory (3.8) we wrote down in the previous subsection for D2-branes stretched between an $O8^-$ plane with no D8-branes and another $O8^-$ plane with 16 D8-branes, since the dilaton and 10-form field are identical in both of these configurations; the only difference is that the D2-D8 fermions are not present, and the boundary conditions on the $O8^+$ plane are different from those on the $O8^-$ plane. In particular, they project the U($2N$) gauge group to USp($2N$) instead of to SO($2N$) (which is another way to see that the rank of the gauge group must be even).

Another naive way to derive this M(atrix) theory would be to start from the effective action of N D0-branes on the Klein bottle, and to perform two Fourier transforms of this action, along the lines of the original derivations of M(atrix) theory compactifications [22, 36]. This analysis is performed in appendix D; it gives the correct boundary conditions, but it only gives the terms in the action coming from the disk and it does not include the effects related to the variation of $z(x^2)$ which come from Möbius strip diagrams, so it leads to an anomalous gauge theory.

The complete action for the M(atrix) theory of M theory on a Klein bottle is

$$S = \frac{1}{4g_{\rm YM}^2}\int dt \int_0^{2\pi R_1} dx^1 \int_0^{\pi R_2} dx^2\, \mathrm{Tr}\Big(-z(x^2)F_{\mu\nu}F^{\mu\nu} + 2z^{1/3}(x^2)(D_\mu Y^j)^2 + \\ + z^{-1/3}(x^2)[Y^j,Y^i][Y^j,Y^i] + 2iz^{1/3}(x^2)\bar{\Psi}_A\gamma^\alpha D_\alpha \Psi_A + \frac{dz^{1/3}(x^2)}{dx^2}\bar{\Psi}_A\Psi_A - \\ -2i\bar{\Psi}_A\gamma^i_{AB}[Y_i,\Psi_B] + \frac{4}{3}\frac{dz(x^2)}{dx^2}\epsilon^{\alpha\beta\gamma}(A_\alpha\partial_\beta A_\gamma + i\frac{2}{3}A_\alpha A_\beta A_\gamma)\Big) \tag{3.11}$$

where, as in the previous subsection,

$$z(x^2) = 1 + \frac{6g_{\rm YM}^2}{\pi}(x^2 - \frac{\pi R_2}{2}). \tag{3.12}$$

The boundary conditions could be derived from the open string theory of D2-branes ending on orientifold planes, but they can also be derived directly in the gauge theory by requiring the absence of boundary terms and consistency with the SUSY transformations which are

[8] The M(atrix) theory of M theory on a Klein bottle was also discussed in [32 – 34] but our results are different. Perhaps some of these other theories arise from different choices of light-like Wilson lines.

[9] The spectrum of D-branes in the $O8^{\pm}$ background was analyzed in [35].

described below. It will be convenient in this section to think of the U($2N$) matrices as made of four $N\times N$ blocks, and to use Pauli matrices that are constant within these blocks. In this notation the scalars Y^i satisfy the boundary conditions[10]

$$\underline{x^2=0}: \qquad Y^j=\sigma^1(Y^j)^T\sigma^1, \qquad \partial_2 Y^j=-\sigma^1(\partial_2 Y^j)^T\sigma^1, \tag{3.13}$$
$$\underline{x^2=\pi R_2}: \qquad Y^j=\sigma^2(Y^j)^T\sigma^2, \qquad \partial_2 Y^j=-\sigma^2(\partial_2 Y^j)^T\sigma^2. \tag{3.14}$$

The SUSY transformations of the action (3.11) are

$$\delta_\epsilon Y^i=-\frac{1}{2}\bar\epsilon_A\gamma^i_{AB}\psi_B, \tag{3.15}$$
$$\delta_\epsilon\psi_A=-\frac{i}{4}z^{-1/3}[Y_i,Y_j]\gamma^{ij}_{AB}\epsilon_B-\frac{1}{4}z^{1/3}F_{\alpha\beta}\gamma^{\alpha\beta}\epsilon_A-\frac{i}{2}D_\alpha Y_i\gamma^\alpha\gamma^i_{AB}\epsilon_B, \tag{3.16}$$
$$\delta_\epsilon A_\alpha=\frac{i}{2}z^{-1/3}\bar\epsilon_A\gamma_\alpha\psi_A. \tag{3.17}$$

Unbroken SUSY transformations are those with $\epsilon_A=i\gamma^2\epsilon_A$. Notice that the SUSY transformations now include the function $z(x^2)$. We can use these transformations to determine the boundary conditions for the fermions. The non-derivative boundary conditions are the naive ones related to (3.13),(3.14), namely

$$\underline{x^2=0}\ :\ \psi_A=-i\sigma^1\gamma^2\psi_A^T\sigma^1, \qquad \underline{x^2=\pi R_2}\ :\ \psi_A=-i\sigma^2\gamma^2\psi_A^T\sigma^2. \tag{3.18}$$

The derivative boundary condition for the upper component of the spinor follows immediately from (3.15),

$$\underline{x^2=0}\ :\ \partial_2\psi_A^+=-\sigma^1(\partial_2\psi_A^+)^T\sigma^1, \qquad \underline{x^2=\pi R_2}\ :\ \partial_2\psi_A^+=-\sigma^2(\partial_2\psi_A^+)^T\sigma^2. \tag{3.19}$$

For the lower component, we have to use (3.16) to obtain

$$\begin{aligned}\underline{x^2=0}: \quad \partial_2\psi_A^-+\frac{1}{3}\frac{z'}{z}\psi_A^- &= \sigma^1\left(\partial_2\psi_A^-+\frac{1}{3}\frac{z'}{z}\psi_A^-\right)^T\sigma^1,\\ \underline{x^2=\pi R_2}: \quad \partial_2\psi_A^-+\frac{1}{3}\frac{z'}{z}\psi_A^- &= \sigma^2\left(\partial_2\psi_A^-+\frac{1}{3}\frac{z'}{z}\psi_A^-\right)^T\sigma^2.\end{aligned} \tag{3.20}$$

The deviations from the naive boundary conditions are proportional to z' which is related to the varying string coupling.

Finally, we will also need to know the boundary conditions on A_2. Note that (3.17) implies (using $\bar\epsilon_A=-i\bar\epsilon_A\gamma^2$)

$$\delta_\epsilon A_2=-\frac{1}{2}z^{-1/3}(y)\bar\epsilon\psi\propto z^{-1/3}(y)\epsilon^-\psi^+\,. \tag{3.21}$$

Thus, the boundary conditions on $(z^{1/3}A_2)$ are the same as those we wrote above for the scalar fields, with no additional terms. Finally, by further investigation of (3.17) one can see that $(z^{1/3}A_{0,1})$ satisfy boundary conditions of exactly the same form (3.20) as ψ^-.

[10] Note that our boundary conditions at $x^2=0$ seem different from those of the previous subsection, but the two are simply related by multiplying all adjoint fields by σ^1.

3.3 The AOA limit of the M(atrix) theory

As we described in the previous section, there is a limit of M theory on a Klein bottle, corresponding to small R_{10}, which gives a weakly coupled string theory — the theory which we called the AOA background. In this limit we should be able to see that the M(atrix) theory we constructed becomes a second quantized theory of strings in this background.[11] Recall that the standard M(atrix) theory for weakly coupled type IIA strings is given by a maximally supersymmetric U(N) $1+1$ dimensional gauge theory; at low energies this flows to a sigma model on $\mathbb{R}^{8N}/S_N$, which describes free type IIA strings (written in Green-Schwarz light-cone gauge) [37 − 40]. The string interactions arise from a twist operator which is the leading, dimension 3, correction to the sigma model action [39]. Similarly, in our case we expect that in the limit where we should obtain a weakly coupled string theory, the low-energy effective action should be a symmetric product of the sigma model of the AOA strings, again written in a Green-Schwarz light-cone gauge (in this gauge the sigma model action is identical to that of AOB strings in a static gauge, just like in the type IIA case we get the action of type IIB strings in a static gauge).

The mapping of parameters described in the previous subsection implies that the limit of small R_{10} corresponds to small R_2 compared to the other scales in the gauge theory. Thus, in this limit we obtain the (strongly coupled limit of the) $1+1$ dimensional theory of the zero modes along the interval. We will analyze this theory in detail for the case of $N=1$, in which the bulk gauge group is U(2); it is easy (by a similar analysis to that of [39]) to see that for higher values of N we obtain (at low energies) the N'th symmetric product of the $N=1$ theories (deformed by higher dimensional operators giving the string interactions).

The zero modes for the scalars Y^i are easily determined by noting that the boundary conditions (3.13), (3.14) are satisfied by the identity matrix

$$Y^i(t,x^1,x^2)_{ab} = Y^i(t,x^1)\mathbb{I}_{ab}, \tag{3.22}$$

where a,b are U(2) indices which we suppress henceforth. The matrices proportional to the identity matrix are actually a completely decoupled sector of the theory (for any value of N), with no interactions. The zero mode analysis for the fermions is a little more involved. The subtlety here is that one should keep in mind that the fermions are Majorana when deriving the equations of motion.[12] To obtain the zero modes we need to solve the equations

$$z^{1/3}\partial_2\psi^+_A = 0, \qquad z^{1/3}\partial_2\psi^-_A + (z^{1/3})'\psi^-_A = 0, \tag{3.23}$$

subject to the boundary conditions described in the previous subsection. For the upper component of the spinor the solution is

$$\psi^+_A(t,x^1,x^2) = \psi^+_A(t,x^1)\mathbb{I}\,,$$

[11] A similar limit for the theory described in section 3.1 should lead to a second quantized theory of $E_8\times E_8$ heterotic strings, but we will not discuss this in detail here.

[12] The relevant part of the Lagrangian in components is $\mathcal{L} \supseteq -z^{1/3}(\psi^+\partial_2\psi^- + \psi^-\partial_2\psi^+) + (z^{1/3})'\psi^-\psi^+$.

which manifestly satisfies the equations of motion and the boundary conditions.[13] For the lower component ψ^-, the equation of motion (3.23) guarantees that the derivative boundary conditions (3.20) are satisfied. In order to satisfy the non-derivative boundary conditions (3.18), we simply choose the direction in the gauge group to be σ^3. Hence, the solution is

$$\psi^-_A(t, x^1, x^2) = \psi^-_A(t, x^1) z^{-1/3}(x^2)\sigma^3 \,. \tag{3.24}$$

Similar zero modes arise for the A_0 and A_1 component of the gauge field. These lead to a U(1) gauge field in the low-energy effective action, but since there are no charged fields, this does not lead to any physical states. Finally, there is a scalar field coming from the zero mode of A_2,

$$A_2(t, x^1, x^2) = A_2(t, x^1) z^{-1/3}(x^2)\mathbb{I} \,. \tag{3.25}$$

This scalar is actually compact due to large gauge transformations, as we describe below.

Of course, all these fields fill out $\mathcal{N} = (0, 8)$ supersymmetry representations in $1 + 1$ dimensions; the vector multiplet contains (A_0, A_1, ψ^-), and the matter multiplet contains (A_2, Y^i, ψ^+). For a detailed description of this kind of SUSY see [27]. For general values of N we find both types of multiplet in the adjoint representation of $\mathrm{U}(N)$, the same field content as in the M(atrix) theory of type IIA strings. The low-energy spectrum turns out to be non-chiral (for any value of N), guaranteeing that there are no anomalies.

In order to identify our theory with the AOA background we need to show that the theory is invariant under the transformation $(-1)^{F_L}$ together with a half-shift on the scalar field coming from A_2. Consider a U(2) gauge transformation of the form

$$g(x) = e^{if(x^2)}\sigma_1 \,. \tag{3.26}$$

Note that our theory is not invariant under generic U(2) gauge transformations since these are broken by the boundary conditions. The transformation (3.26) preserves all the boundary conditions. In order for it to leave the theory in the same topological sector (the simplest way to verify this is to regard the cylinder as an orbifold of the torus and use the usual classification of sectors on the torus) we require that $f(\pi R_2) - f(0) = (2n+1)\pi/2$ for some integer n. In general the transformation (3.26) mixes the zero mode (3.25) with other modes of A_2; however, all other modes can be gauged away so this mixing is not really physical. We can work in a gauge where all the non-zero modes are set to zero, and an appropriate choice of a large gauge transformation which preserves this gauge is

$$f(x^2) = \frac{z^{2/3}(x^2)\pi/2}{z^{2/3}(\pi R_2) - z^{2/3}(0)} \quad \rightarrow \quad \partial_2 f(x^2) = \frac{z'}{3}\frac{z^{-1/3}(x^2)\pi}{z^{2/3}(\pi R_2) - z^{2/3}(0)} \,. \tag{3.27}$$

The action of this transformation on the zero mode (3.25) implies that we should identify

$$A_2(t, x^1) \simeq A_2(t, x^1) + \frac{1}{3}\frac{z'\pi}{z^{2/3}(\pi R_2) - z^{2/3}(0)} = A_2 + \frac{2g^2_{\mathrm{YM}}}{z^{2/3}(\pi R_2) - z^{2/3}(0)}. \tag{3.28}$$

[13]The existence of this mode is guaranteed by the fact that it is actually the Goldstino for the 8 supercharges broken by the D2-branes.

In the low-energy effective action, the large gauge transformation (3.26) acts also on the vector multiplet, implying that the identification (3.28) is accompanied by

$$\psi^-(t,x^1) \to -\psi^-(t,x^1), \qquad A_{0,1}(t,x^1) \to -A_{0,1}(t,x^1). \tag{3.29}$$

This establishes that our low-energy $1+1$ dimensional sigma model is gauged by $((-)^{F_L} \times$ shift), as expected.

Next, we wish to compute the physical radius of the scalar arising from A_2 to verify that it agrees with the physical radius we expect. Carrying out the dimensional reduction explicitly (setting all other fields except the zero mode of A_2 to zero) we get

$$\begin{aligned} S &= \frac{1}{4g_{\rm YM}^2}\int d^2x \int_0^{\pi R_2} dx^2 {\rm Tr}_f\Big(-z(x^2)F_{\mu\nu}F^{\mu\nu}+\dots\Big) \\ &= \frac{1}{g_{\rm YM}^2}\int d^2x \int_0^{\pi R_2} dx^2 (z^{1/3}(x^2)\partial_\mu A_2 \partial^\mu A_2 + \dots) \\ &= \frac{1}{g_{\rm YM}^2}\int d^2x (\partial_\mu A_2 \partial^\mu A_2) \int_0^{\pi R_2} dx^2 z^{1/3}(x^2) \\ &= \frac{\pi}{8g_{\rm YM}^4}\int d^2x (\partial_\mu A_2 \partial^\mu A_2)(z^{4/3}(\pi R_2) - z^{4/3}(0)). \end{aligned} \tag{3.30}$$

Thus, the physical, dimensionless radius of the scalar A_2 is

$$\begin{aligned} &\frac{1}{2\pi}\cdot\frac{2g_{\rm YM}^2}{z^{2/3}(\pi R_2)-z^{2/3}(0)}\cdot\sqrt{\frac{\pi}{8g_{\rm YM}^4}(z^{4/3}(\pi R_2)-z^{4/3}(0))} \\ &\qquad = \frac{1}{z^{2/3}(\pi R_2)-z^{2/3}(0)}\sqrt{\frac{1}{8\pi}(z^{4/3}(\pi R_2)-z^{4/3}(0))} \\ &\qquad = \sqrt{\frac{1}{8\pi}}\sqrt{\frac{z^{2/3}(\pi R_2)+z^{2/3}(0)}{z^{2/3}(\pi R_2)-z^{2/3}(0)}}\,. \end{aligned} \tag{3.31}$$

Recall that, as discussed in the previous section, the AOB sigma model has a self-T-duality at a physical radius of $(8\pi)^{-1/2}$. We see from (3.31) that this corresponds to $z(0) = 0$, which is exactly the case where the Yang-Mills coupling diverges at one side of the interval (due to a diverging coupling on the $O8^-$ plane in the $O8^\pm$ background). As discussed in the previous section, at this point of diverging coupling the $O8^\pm$ background has an enhanced SU(2) gauge symmetry in space-time, which should correspond to an enhanced SU(2) global symmetry in our gauge theory; we see that in the low-energy effective action this enhanced global symmetry is precisely the one associated with the AOB sigma model at the self-dual radius.

In the M(atrix) theory interpretation of our gauge theory, the line $z(0) = 0$ precisely maps to the line of enhanced SU(2) symmetry of the compactification of M theory on a Klein bottle. Note that our gauge theory only makes sense for $z(0) \geq 0$, since otherwise we obtain negative kinetic terms for some fields. Thus, our M(atrix) theory description only makes sense above the self-dual line in figure 2. Of course, the theories below the line are identified by a duality with the theories above the line, so we do have a valid description for the full moduli space of Klein bottle compactifications. A similar analysis for the case of a cylinder (with no Wilson lines) again shows that infinite gauge coupling is obtained precisely on the line of enhanced SU(2) symmetry in space-time shown in figure 1.

4. Conclusions and open questions

In this paper we analyzed in detail the moduli space of nine dimensional compactifications of M theory with $\mathcal{N} = 1$ supersymmetry and their M(atrix) theory descriptions. We found several surprises: the moduli space of theories with rank 2 turned out to have two disconnected components, and in order to obtain a consistent description of theories with cross-caps we had to conjecture a non-perturbative splitting of $O8^0$ planes into a D8-brane and an infinitely coupled $O8^{(-1)}$ plane. The M(atrix) theories we found are $2+1$ dimensional gauge theories on a cylinder, but generically they are rather complicated theories with a varying gauge coupling. The only case where we obtained a standard gauge theory is the case of M theory on a cylinder with a Wilson line breaking the gauge theory to SO(16) $\times$ SO(16); the simplicity of the M(atrix) theory in this case is related to the fact that (unlike all other backgrounds we discussed) this background does not have a subspace with enhanced gauge symmetry in space-time. In all other cases, the manifold with enhanced gauge symmetry in space-time is mapped in M(atrix) theory to having infinite gauge coupling on one side of the interval.

There are several interesting directions for further research. We provided some circumstantial evidence for our description of the X and $1/2X$ backgrounds, but it would be nice to obtain more evidence for this. One way to obtain such evidence would be to compactify these backgrounds on an additional circle; these backgrounds then have an F theory description, with the $O8^+$ plane becoming an unresolvable D_8 singularity [15]. One can then consider the nine dimensional limit of this background, as discussed in [41, 42], and hope to recover our picture with the D8-brane emitted into the bulk. Another possible way to study the X background is by its M(atrix) theory dual; this is given by the limit of the M(atrix) theory for the Klein bottle that we constructed in section 3 in which the circle is much smaller than the interval. It would be nice to understand the dynamics of the theory in this limit in detail in order to understand the X background better; it may be necessary for this to analyze the regime of large N with energies of order $1/N$, which is most directly related to the space-time physics.

Another possible way to study these backgrounds is by brane probes, such as D2-branes stretched between the two orientifold planes. These D2-branes are interesting also for another reason, since (at least naively) they provide the M(atrix) theory for some of the additional nine dimensional backgrounds that we did not discuss in the previous section, with the M(atrix) theory for the DP background (and the other backgrounds in the same moduli space) related to D2-branes in the X background, and the M(atrix) theory for M theory on a Möbius strip related to D2-branes in the $1/2X$ background. It would be interesting to understand these theories better; naively one obtains an anomaly from the D2-D8 fermions, and it is not clear how this is cancelled.

Another natural question involves the realization of the non-perturbative duality symmetries in the M(atrix) theory. In toroidal compactifications of M theory, such dualities are related [43] to non-trivial dualities relating electric and magnetic fields in the M(atrix) theory gauge theories. We mentioned above that at the enhanced SU(2) symmetry points in space-time the M(atrix) theory is supposed to have a non-trivial enhanced SU(2) global symmetry (which can also be seen by viewing this theory as the theory of a D2-brane in a type I' background with an enhanced symmetry); the realization of this symmetry will be discussed in detail in [44]. More generally, there is a $\mathbb{Z}_2$ duality symmetry relating the two sides of each graph in figures 1 and 2, which should map to a duality between two gauge theories in M(atrix) theory. Unfortunately, so far we have only been able to find the M(atrix) description for the large radius region of the moduli space (as described above), and we could not yet find an independent (dual) description for the small radius region. It would be interesting to investigate this further [44]; it requires continuing the moduli space of backgrounds with $O8^-$ planes beyond the point where the string coupling diverges at one of the orientifold planes, but without changing to the dual variables.

Finally, it would be interesting to generalize our results concerning the classification of backgrounds with 16 supercharges to lower dimensions (for a partial classification see [1]), and to see if there are any new components or unexplored corners of the moduli space there (as we found in nine dimensions).

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A. Spinor conventions

We summarize our conventions for spinors in $2+1$ dimensions. We choose the metric to be mostly minus, $\eta^{\alpha\beta} = \mathrm{diag}(1,-1,-1)$. The gamma matrices of SO(2,1) satisfy $\{\gamma^\alpha,\gamma^\beta\} = 2\eta^{\alpha\beta}$, where α,β are vector indices of SO(2,1), and the spinor indices are denoted by $a,b=1,2$ (but are usually suppressed). A convenient basis for SO(2,1) gamma matrices is

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{A.1}$$

Our fermions lie in spinor representations of the global $\mathrm{Spin}(7)_R$ symmetry. The gamma matrices of $\mathrm{Spin}(7)_R$ satisfy $\{\gamma^i,\gamma^j\} = 2\delta^{ij}$, where i,j are vector indices of $\mathrm{Spin}(7)_R$. We shall denote spinor indices of $\mathrm{Spin}(7)_R$ by A,B. Finally, for a Majorana fermion ψ, we define $\bar\psi = \psi^T\gamma^0$.

B. Periodicities of p-form fields in some 9d compactifications

In this appendix we discuss the periodicities of the p-form fields in some of the backgrounds described in section 2.2. We being with the AOA background [12, 13], in which one gauges the following symmetry of type IIA string theory:

$$(-)^{F_L} \times (X^9 \to X^9 + 2\pi R_9). \tag{B.1}$$

The NS-NS 2-form $B_{\mu\nu}$ is periodic along the circle in the 9'th direction:

$$B_{\mu\nu}(x^9 + 2\pi R_9) = B_{\mu\nu}(x^9) \tag{B.2}$$

and leads to a massless mode. On the other hand, the RR 3-form field $A^{(3)}$ gets a minus sign from $(-)^{F_L}$, hence it has to be antiperiodic in the direction x^9. In components,

$$A^{(3)}_{\mu\nu\rho}(x^9 + 2\pi R_9) = -A^{(3)}_{\mu\nu\rho}(x^9). \tag{B.3}$$

As discussed in section 2.2, the target space of the M theory lift of this background is $\mathbb{R}^{8,1} \times K2$, with $x^{10} \to -x^{10}$ under the involution. In eleven dimensions the involution must take $C^{M(3)} \to -C^{M(3)}$ in order to be a symmetry of M theory. We can also verify that this is consistent with the periodicities we wrote above. For $C^{M(3)}_{\mu\nu(10)}$, taking into account the coordinate transformation we get that

$$C^{M(3)}_{\mu\nu(10)}(x^9 + 2\pi R_9, -x^{10}) = C^{M(3)}_{\mu\nu(10)}(x^9, x^{10}). \tag{B.4}$$

Reducing this on x^{10} we find agreement with (B.2). The other components are all anti-periodic, in agreement with (B.3).

Next, we compactify this M theory on an additional circle $\mathbb{R}^{7,1} \times K2 \times S^1$. We can reduce on this S^1 to obtain a different IIA theory, which is the one described by (2.6). Now the fact that $C^{M(3)} \to -C^{M(3)}$ implies different periodicities in the type IIA theory. For the NS-NS 2-form

$$B^{(2)} \to -B^{(2)}, \tag{B.5}$$

which in components implies for instance

$$B^{(2)}_{\mu 8}(-x^8, x^9 + 2\pi R_9) = B^{(2)}_{\mu 8}(x^8, x^9). \tag{B.6}$$

Similarly, for the RR 3-form,

$$A^{(3)} \to -A^{(3)}. \tag{B.7}$$

These results can also be derived directly from the IIA perspective. We first note that $\Omega \times (X^8 \to -X^8)$ is a symmetry of IIA because it changes the chirality of fermions and the RR forms twice.[14] Since Ω flips its sign the B field has to be odd under the involution, in agreement with (B.5). We can check the consistency on the RR sector in the Green-Schwarz formalism. In this formalism type IIA superstrings have space-time fermions $\theta_L(z)$ on the

[14]By super-conformal invariance of the worldsheet we get that the worldsheet fermions ψ and $\widetilde{\psi}$ flip sign under this involution and hence change the chirality in the R sectors.

left and $\theta_R(\bar{z})$ on the right. The transformation (2.6) that produces the Klein bottle acts on the space-time fermions as

$$\theta_L(z) \to \Gamma^8 \theta_R(\bar{z}), \tag{B.8}$$
$$\theta_R(\bar{z}) \to \Gamma^8 \theta_L(z). \tag{B.9}$$

The RR vertex operator is

$$V = \theta_R(\bar{z})^\dagger \Gamma_{\mu_1 \cdots \mu_p} \theta_L(z) + \text{c.c.}. \tag{B.10}$$

Under the transformation the fermionic part becomes

$$\begin{aligned} \theta_L(z)^\dagger \Gamma^8 \Gamma_{\mu_1 \cdots \mu_p} \Gamma^8 \theta_R(\bar{z}) + \text{c.c.} &= -\theta_R(\bar{z})^T (\Gamma^8)^T (\Gamma_{\mu_1 \cdots \mu_p})^T (\Gamma^8)^T \theta_L(z)^* + \text{c.c.} \\ &= -\theta_R(\bar{z})^\dagger \Gamma^8 (\Gamma_{\mu_1 \cdots \mu_p})^\dagger \Gamma^8 \theta_L(z) + \text{c.c.} \\ &= (-1)^{[p/2]+1} \theta_R(\bar{z})^\dagger \Gamma^8 \Gamma_{\mu_1 \cdots \mu_p} \Gamma^8 \theta_L(z) + \text{c.c.}. \end{aligned} \tag{B.11}$$

Thus, the 1-form transforms as a 1-form (changing sign only for the component in the 8 direction) and the 3-form transforms as a pseudo-3-form (changing sign for components not in the 8 direction). In other words, only the twisted 3-form cohomology survives, and only the untwisted cohomology of 1-forms survives (see [45] for more details[15]).

C. Modular invariance and T-duality in the AOA partition function

In this appendix we compute the partition function of the AOA theory (defined in section 2) and verify that it is modular invariant and T-duality invariant. Throughout this appendix we set $l_s = 1$. The subtle point is that the definition of the theory includes special phases in the twisted sectors (which are required for the consistency of the theory).

The sectors of the theory and their matter GSO projections (including $(-1)^w$) are [13]:[16]

A: $NS - NS$ sector with $n \in \mathbb{Z}, w \in 2\mathbb{Z}$ and GSO projection $-/-$.

B: $NS - R$ sector with $n \in \mathbb{Z}, w \in 2\mathbb{Z}$ and GSO projection $-/+$.

C: $R - NS$ sector with $n \in \mathbb{Z} + \frac{1}{2}, w \in 2\mathbb{Z}$ and GSO projection $-/-$.

D: $R - R$ sector with $n \in \mathbb{Z} + \frac{1}{2}, w \in 2\mathbb{Z}$ and GSO projection $-/+$.

E: $R - NS$ sector with $n \in \mathbb{Z}, w \in 2\mathbb{Z} + 1$ and GSO projection $+/-$.

F: $R - R$ sector with $n \in \mathbb{Z}, w \in 2\mathbb{Z} + 1$ and GSO projection $+/+$.

G: $NS - NS$ sector with $n \in \mathbb{Z} + \frac{1}{2}, w \in 2\mathbb{Z} + 1$ and GSO projection $+/-$.

[15] Loosely speaking, the untwisted cohomology is the subset of the covering space cohomology which contains only the forms which are even. The twisted cohomology contains the odd forms.

[16] Note that the conventions in [13] for right/left-moving sectors are different from the usual.

H: $NS - R$ sector with $n \in \mathbb{Z} + \frac{1}{2}, w \in 2\mathbb{Z}+1$ and GSO projection $+/+$.

Let Z^α_β be a path integral on the torus over fermion fields ψ with periodicities

$$\begin{aligned}\psi(w+2\pi) &= -e^{i\pi\alpha}\psi(w),\\ \psi(w+2\pi\tau) &= -e^{i\pi\beta}\psi(w).\end{aligned} \tag{C.1}$$

The fermionic partition sums are then ($0, v, s, c$ stand for the four possible combinations of NS and R sectors with GSO projections)

$$\begin{aligned}\chi_0 &= \frac{1}{2}\big[Z^0_0(\tau)^4 + Z^0_1(\tau)^4\big],\\ \chi_v &= \frac{1}{2}\big[Z^0_0(\tau)^4 - Z^0_1(\tau)^4\big],\\ \chi_s &= \frac{1}{2}\big[Z^1_0(\tau)^4 + Z^1_1(\tau)^4\big],\\ \chi_c &= \frac{1}{2}\big[Z^1_0(\tau)^4 - Z^1_1(\tau)^4\big].\end{aligned} \tag{C.2}$$

The partition function on the torus that we obtain by summing over all the sectors described above is:

$$\begin{aligned}Z(\tau) = iV_{10}\int_F \frac{d^2\tau}{16\pi^2\tau_2^2} Z^7_X |\eta(\tau)|^{-2}(\chi_v - \chi_s)\,\times \qquad\qquad (C.3)\\ \times\Bigg(\bar{\chi}_v \sum_{n\in\mathbb{Z}, w\in 2\mathbb{Z}} \exp\big[-\pi\tau_2(\frac{n^2}{R^2} + w^2R^2) + 2\pi i\tau_1 nw\big]\\ +\bar{\chi}_0 \sum_{n\in\mathbb{Z}+1/2, w\in 2\mathbb{Z}+1} \exp\big[-\pi\tau_2(\frac{n^2}{R^2} + w^2R^2) + 2\pi i\tau_1 nw\big]\\ -\bar{\chi}_s \sum_{n\in\mathbb{Z}, w\in 2\mathbb{Z}+1} \exp\big[-\pi\tau_2(\frac{n^2}{R^2} + w^2R^2) + 2\pi i\tau_1 nw\big]\\ -\bar{\chi}_c \sum_{n\in\mathbb{Z}+1/2, w\in 2\mathbb{Z}} \exp\big[-\pi\tau_2(\frac{n^2}{R^2} + w^2R^2) + 2\pi i\tau_1 nw\big]\Bigg),\end{aligned}$$

where Z^7_X is the partition sum for the 7 transverse bosons. After resummation we get

$$\begin{aligned}Z(\tau) = iV_{10}\int_F \frac{d^2\tau}{16\pi^2\tau_2^2} Z^8_X(\chi_v - \chi_s) \sum_{m,w\in\mathbb{Z}} \Bigg(\bar{\chi}_v \exp\left[-\frac{\pi R^2|m - 2w\tau|^2}{\tau_2}\right] \qquad (C.4)\\ +\bar{\chi}_0 \exp\left[-\frac{\pi R^2|m-(2w+1)\tau|^2}{\tau_2} + \pi i m\right]\\ -\bar{\chi}_s \exp\left[-\frac{\pi R^2|m-(2w+1)\tau|^2}{\tau_2}\right]\\ -\bar{\chi}_c \exp\left[-\frac{\pi R^2|m-2w\tau|^2}{\tau_2} + \pi i m\right]\Bigg).\end{aligned}$$

From the modular transformations of Z_β^α:

$$
\begin{aligned}
Z_\beta^\alpha(\tau+1) &= \epsilon^{\pi i(3\alpha^2-1)/12} Z_{\beta-\alpha+1}^\alpha(\tau),\\
Z_\beta^\alpha(-1/\tau) &= Z_\alpha^{-\beta},
\end{aligned}
\tag{C.5}
$$

we get the transformations of the χ's:

$$
\begin{aligned}
\chi_v(\tau+1) &= \epsilon^{2\pi i/3}\chi_v(\tau),\\
\chi_0(\tau+1) &= \epsilon^{-\pi i/3}\chi_0(\tau),\\
\chi_{s/c}(\tau+1) &= \epsilon^{2\pi i/3}\chi_{s/c}(\tau),\\
\chi_{v/0}(-1/\tau) &= \frac{1}{2}\big[Z_0^0(\tau)^4 \mp Z_0^1(\tau)^4\big],\\
\chi_{s/c}(-1/\tau) &= \frac{1}{2}\big[Z_1^0(\tau)^4 \pm Z_1^1(\tau)^4\big].
\end{aligned}
\tag{C.6}
$$

Modular invariance of (C.4) under $\tau \to \tau+1$ is straightforward: the phase from $(\chi_v - \chi_s)$ is cancelled by the phase coming from the left-moving sector (the phase of $\bar{\chi}_0$ gets another (-1) from the transformation of the sum over momenta and windings). The modular invariance under $\tau \to -\frac{1}{\tau}$ follows from the fact that if we take $m \to -w$ and $w \to m$, the bosonic sums that multiply $(\bar{Z_0^0})^4$ and $(\bar{Z_1^1})^4$ do not change and the bosonic sums that multiply $(\bar{Z_1^0})^4$ and $(\bar{Z_0^1})^4$ are switched.

If we look at the partition function (C.3), we see that it is invariant under T-duality with $R \to \frac{1}{2R}$, $n \to \frac{w}{2}$, and $w \to 2n$. Recall that in order to verify T-duality one has to flip the GSO projection of the right-moving modes of the Ramond sector, and then the partition function is manifestly invariant (as usual, to verify this one needs to use $Z_1^1 = 0$).

D. Quantum mechanics of D0-branes on the Klein bottle

In this appendix we take a naive approach to the derivation of M(atrix) theory for M theory on a Klein bottle, by starting with D0-branes on a Klein bottle, imposing the appropriate identifications on the Chan-Paton indices, and performing a Fourier transform. Our D0-branes are moving in the background with the identification (2.6), but we will rename the directions of the Klein bottle x^1 and x^2, and denote the periodicities of these variables by $2\pi R_1'$ and $4\pi R_2'$. Our identifications of the Chan-Paton indices will differ from those in [33] (which describe a non-commutative version, as pointed out in [46]), but they are similar to those of [34]. We construct the identifications by simply moving strings with the transformation

$$
(x^1, x^2) \sim (-x^1, x^2 + 2\pi R_2') \tag{D.1}
$$

and then reversing their orientation. Figure 7 is an example of how we obtain the various relations.

For convenience, we should now have $2N$ D0-branes in each fundamental domain of the torus. Let X^i stand for all the transverse Euclidean directions to the Klein bottle ($i = 3, \cdots, 9$). We will write everything in components first and then try to find a more elegant formalism. In general, the indices are $\Lambda_{k,l;a,b;i,j}$ where k and l are two dimensional

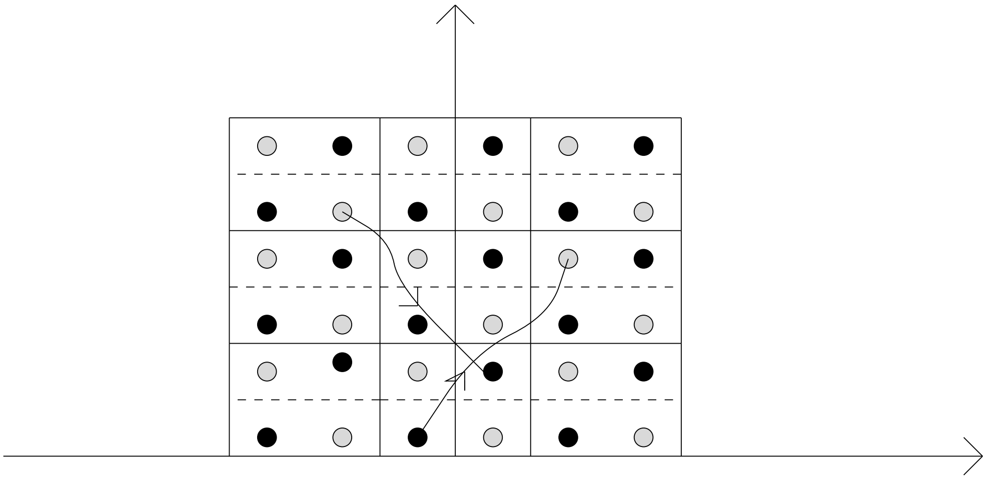

Figure 7: To demonstrate the identifications we propose, we show an explicit example of how it works. In this figure, the middle cell in the lowest row is the (0,0) cell. We act with the orientifold transformation to demonstrate the following equality $X_{(0,0),(1,1);1,2;i,j} = X_{(-1,2),(0,0);1,2;j,i} = X_{(0,0),(1,-2);1,2;j,i}$ where X is any of the orthogonal coordinates. The dashed line separates each torus fundamental domain to two copies of the Klein bottle. The filled circles are D0-branes (and the two colors correspond to two different D0-branes). The two depicted strings are identified under the transformation.

vectors of integers (corresponding to the periodic images of the D0-branes), a and b are in $\{1,2\}$ and i and j define an $N \times N$ matrix. The identifications coming from the torus periodicities are the standard ones (see, e.g., [36]). For our additional identifications, define $\hat{k} = (-k_1, k_2)$, and then:

$$\begin{aligned} X^i_{k,l;1,1;i,j} &= X^i_{\hat{l},\hat{k};2,2;j,i}, \\ X^i_{k,l;2,2;i,j} &= X^i_{\hat{l},\hat{k};1,1;j,i} = X^i_{\hat{l}+e_2,\hat{k}+e_2;1,1;j,i}, \\ X^i_{k,l;1,2;i,j} &= X^i_{\hat{l}+e_2,\hat{k};1,2;j,i}, \\ X^i_{k,l;2,1;i,j} &= X^i_{\hat{l},\hat{k}+e_2;2,1;j,i}. \end{aligned} \tag{D.2}$$

These are completely geometric identifications, similar to the particular one exhibited in figure 7. The purpose of writing the second row as it is will become clear below. One should also notice that there are no identifications on the orthogonal coordinates besides (D.2) and the usual torus identifications.

On the A^0 matrices similar identifications hold up to an additional minus sign because of the action on the vertex operator (which contains a normal derivative):

$$\begin{aligned} A^0_{k,l;1,1;i,j} &= -A^0_{\hat{l},\hat{k};2,2;j,i}, \\ A^0_{k,l;2,2;i,j} &= -A^0_{\hat{l},\hat{k};1,1;j,i} = -A^0_{\hat{l}+e_2,\hat{k}+e_2;1,1;j,i}, \\ A^0_{k,l;1,2;i,j} &= -A^0_{\hat{l}+e_2,\hat{k};1,2;j,i}, \\ A^0_{k,l;2,1;i,j} &= -A^0_{\hat{l},\hat{k}+e_2;2,1;j,i}. \end{aligned} \tag{D.3}$$

The constraints on the coordinates X^1 and X^2 are also very easy to find by following the fate of a string after the symmetry operation acts on it. The results are summarized in the following list:

$$\begin{aligned}
X^1_{k,l;1,1;i,j} &= -X^1_{\hat{l},\hat{k};2,2;j,i}, \\
X^1_{k,l;2,2;i,j} &= -X^1_{\hat{l},\hat{k};1,1;j,i} = -X^1_{\hat{l}+e_2,\hat{k}+e_2;1,1;j,i}, \\
X^1_{k,l;1,2;i,j} &= -X^1_{\hat{l}+e_2,\hat{k};1,2;j,i}, \\
X^1_{k,l;2,1;i,j} &= -X^1_{\hat{l},\hat{k}+e_2;2,1;j,i}, \\
X^2_{k,l;1,1;i,j} &= X^2_{\hat{l},\hat{k};2,2;j,i} - \delta_{l,k} 2\pi R'_2 \delta_{i,j}, \\
X^2_{k,l;2,2;i,j} &= X^2_{\hat{l},\hat{k};1,1;j,i} + \delta_{l,k} 2\pi R'_2 \delta_{i,j} = X^2_{\hat{l}+e_2,\hat{k}+e_2;1,1;j,i} - \delta_{l,k} 2\pi R'_2 \delta_{i,j}, \\
X^2_{k,l;1,2;i,j} &= X^2_{\hat{l}+e_2,\hat{k};1,2;j,i}, \\
X^2_{k,l;2,1;i,j} &= X^2_{\hat{l},\hat{k}+e_2;2,1;j,i}.
\end{aligned} \tag{D.4}$$

As was explained in section 3.2, we know there exists a nice T-dual description of this theory as the worldvolume of a D2-brane. It should be possible, therefore, to rearrange the very inconvenient set of identifications on the infinite matrices that we wrote above as a $2+1$ dimensional gauge theory of finite matrices.

To see this, define

$$M_{k,k';a,a'} = \delta_{k',\hat{k}+\tau_{a,a'} e_2} \sigma^1_{a,a'}, \tag{D.5}$$

where we have defined

$$\tau = \frac{1}{2}\left(\sigma^1 - i\sigma^2\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{D.6}$$

One can easily show that

$$(M^{-1})_{k',l;a',a} = \delta_{k',\hat{l}+\tau_{a,a'} e_2} \sigma^1_{a',a}. \tag{D.7}$$

Now, equations (D.2), (D.3), (D.4) can be neatly rewritten in the following form:

$$\begin{aligned}
X^i &= M(X^i)^T M^{-1}, \\
A^0 &= -M(A^0)^T M^{-1}, \\
X^1 &= -M(X^1)^T M^{-1}, \\
X^2 &= M(X^2)^T M^{-1} - 2\pi R'_2 \delta_{k,l} \delta_{a,b} \delta_{i,j}.
\end{aligned} \tag{D.8}$$

Of course, these identifications are imposed together with the torus identifications. The equivalence of (D.8) with (D.2), (D.3), (D.4) can be demonstrated by simple calculations.

Next, we rearrange the fields as Fourier components in the k,l indices:

$$
\begin{aligned}
A^0_{a,b;i,j}(x,t) &= \sum_k (A^0)_{0,k;a,b;i,j} e^{i\left(\frac{k_1 x_1}{R_1}+\frac{k_2 x_2}{R_2}\right)}, \\
A^1_{a,b;i,j}(x,t) &= \sum_k X^1_{0,k;a,b;i,j} e^{i\left(\frac{k_1 x_1}{R_1}+\frac{k_2 x_2}{R_2}\right)}, \\
A^2_{a,b;i,j}(x,t) &= \sum_k X^2_{0,k;a,b;i,j} e^{i\left(\frac{k_1 x_1}{R_1}+\frac{k_2 x_2}{R_2}\right)}, \\
\forall l = 3,\ldots,9 \qquad X^l_{a,b;i,j}(x,t) &= \sum_k X^l_{0,k;a,b;i,j} e^{i\left(\frac{k_1 x_1}{R_1}+\frac{k_2 x_2}{R_2}\right)},
\end{aligned} \tag{D.9}
$$

where $R_1 \equiv l_s^2/R_1'$ and $R_2 \equiv l_s^2/2R_2'$. The Lagrangian governing these fields is given by a maximally supersymmetric (after including the fermions) Yang-Mills theory in $2+1$ dimensions (16 real supercharges), which is the dimensional reduction of the ten dimensional $\mathcal{N}=1$ SYM theory. However, there are some inter-relations among the fields which we now derive. For convenience we suppress the i,j indices. The inter-relations comprise the following set of additional relations on the theory with 16 supercharges ($j=3,\cdots,9$):

$$
\begin{aligned}
&X^j_{1,1}(x,t) = (X^j_{2,2}(-\hat{x},t))^T, \\
&X^j_{1,2}(x,t) = e^{-i\frac{x_2}{R_2}}(X^j_{1,2}(-\hat{x},t))^T, \\
&X^j_{2,1}(x,t) = e^{i\frac{x_2}{R_2}}(X^j_{2,1}(-\hat{x},t))^T, \\
&A^0_{1,1}(x,t) = -(A^0_{2,2}(-\hat{x},t))^T, \\
&A^0_{1,2}(x,t) = -e^{-i\frac{x_2}{R_2}}(A^0_{1,2}(-\hat{x},t))^T, \\
&A^0_{2,1}(x,t) = -e^{i\frac{x_2}{R_2}}(A^0_{2,1}(-\hat{x},t))^T, \\
&A^1_{1,1}(x,t) = -(A^1_{2,2}(-\hat{x},t))^T, \\
&A^1_{1,2}(x,t) = -e^{-i\frac{x_2}{R_2}}(A^1_{1,2}(-\hat{x},t))^T, \\
&A^1_{2,1}(x,t) = -e^{i\frac{x_2}{R_2}}(A^1_{2,1}(-\hat{x},t))^T, \\
&A^2_{1,1}(x,t) = (A^2_{2,2}(-\hat{x},t))^T - 2\pi R_2'\delta_{i,j}, \\
&A^2_{1,2}(x,t) = e^{-i\frac{x_2}{R_2}}(A^2_{1,2}(-\hat{x},t))^T, \\
&A^2_{2,1}(x,t) = e^{i\frac{x_2}{R_2}}(A^2_{2,1}(-\hat{x},t))^T.
\end{aligned} \tag{D.10}
$$

We see that our theory can be written on a cylinder of volume $(2\pi R_1)\times(\pi R_2)$ with orientifold planes at the boundaries. We can read the relevant boundary conditions from (D.10). As the action appears to be an orbifold of the maximally supersymmetric action we write it explicitly:

$$
S = \frac{1}{2g^2_{\mathrm{YM}}}\int_{\mathbb{T}^2} dt d^2x \mathrm{Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (D_\mu X^j)^2 + \frac{1}{2}[X^j,X^i][X^j,X^i] + \text{fermions}\right). \tag{D.11}
$$

The size of $\mathbb{T}^2$ is $(2\pi R_1)\times(2\pi R_2)$. All the fields are periodic and in our case take the form

$$X = \begin{pmatrix} X_{1,1;i,j} & X_{1,2;i,j} \\ X_{2,1;i,j} & X_{2,2;i,j} \end{pmatrix}, \qquad A = \begin{pmatrix} A_{1,1;i,j} & A_{1,2;i,j} \\ A_{2,1;i,j} & A_{2,2;i,j} \end{pmatrix}, \qquad i,j = 1,\cdots,N. \tag{D.12}$$

We split the action:

$$\begin{aligned} S = & \frac{1}{2g_{\mathrm{YM}}^2}\int dt \int_0^{2\pi R_1} dx^1 \int_{-\pi R_2}^{0} dx^2 \,\mathrm{Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (D_\mu X^j)^2 + \frac{1}{2}[X^j,X^i][X^j,X^i]\right) \\ & + \frac{1}{2g_{\mathrm{YM}}^2}\int dt \int_0^{2\pi R_1} dx^1 \int_0^{\pi R_2} dx^2 \,\mathrm{Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (D_\mu X^j)^2 + \frac{1}{2}[X^j,X^i][X^j,X^i]\right) \end{aligned} \tag{D.13}$$

Using (D.10) we can reexpress all the fields in the domain $[-\pi R_2, 0]$ as fields in the domain $[0,\pi R_2]$. The computation becomes very easy if one notices that the set of equations (D.10) is actually simple, $\forall j = 3,\cdots,9$:

$$\begin{aligned} X^j(x,t) &= e^{-\frac{ix^2}{2R_2}\sigma^3}\sigma^1 (X^j(-\hat{x},t))^T \sigma^1 e^{\frac{ix^2}{2R_2}\sigma^3}, \\ A^{0(1)}(x,t) &= -e^{-\frac{ix^2}{2R_2}\sigma^3}\sigma^1 (A^{0(1)}(-\hat{x},t))^T \sigma^1 e^{\frac{ix^2}{2R_2}\sigma^3}, \\ A^2(x,t) &= e^{-\frac{ix^2}{2R_2}\sigma^3}\sigma^1 (A^2(-\hat{x},t))^T \sigma^1 e^{\frac{ix^2}{2R_2}\sigma^3} - 2\pi R_2'\sigma^3, \end{aligned} \tag{D.14}$$

where the relations are on the full $2N\times 2N$ matrices. The sigma matrices are always tensored with the unit matrix of dimension $N\times N$. One can see that this gives a nice Lagrangian, as most of the commutators are invariant, or change sign. One easy way to proceed is to rewrite the above lagrangian with new fields in the following way:

$$\begin{aligned} \widetilde{X}^j(x,t) &= e^{\frac{ix^2\sigma^3}{4R_2}} X^j(x,t) e^{-\frac{ix^2\sigma^3}{4R_2}}, \\ \widetilde{A}^{0(1)}(x,t) &= e^{\frac{ix^2\sigma^3}{4R_2}} A^{0(1)}(x,t) e^{-\frac{ix^2\sigma^3}{4R_2}}, \\ \widetilde{A}^2(x,t) &= e^{\frac{ix^2\sigma^3}{4R_2}} A^2(x,t) e^{-\frac{ix^2\sigma^3}{4R_2}} + \sigma^3\pi R_2'. \end{aligned} \tag{D.15}$$

This transformation is a $\mathrm{U}(2N)$ gauge transformation. Hence the Lagrangian of the tilded fields is actually the same Lagrangian as (D.13). We will still name those fields in the usual notation omitting the tildes. The identification of the tilded fields[17] is very suggestive:

$$\begin{aligned} X^j(x,t) &= \sigma^1 (X^j(-\hat{x},t))^T \sigma^1, \\ A^{0(1)}(x,t) &= -\sigma^1 (A^{0(1)}(-\hat{x},t))^T \sigma^1, \\ A^2(x,t) &= \sigma^1 (A^2(-\hat{x},t))^T \sigma^1. \end{aligned} \tag{D.16}$$

Using this transformations we transform the $[-\pi R_2, 0]$ interval of the action to $[0,\pi R_2]$, and we find that it precisely agrees with the original action for this interval. The relative

[17] As mentioned, we won't use the notation of the tildes anymore.

minus signs in the gauge fields exactly compensate for the minus signs that arise because $[A^T, B^T] = -([A,B])^T$ and because $\partial_2 \to -\partial_2$. So, our final result is

$$S = \frac{1}{2g_{\rm YM}^2} \int dt \int_0^{2\pi R_1} dx^1 \int_0^{\pi R_2} dx^2 \, {\rm Tr} \Big(-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + (D_\mu X^j)^2 + \\ + \frac{1}{2} [X^j, X^i][X^j, X^i] + \text{fermions} \Big). \tag{D.17}$$

We will determine $g_{\rm YM}$ soon, so we omitted the factor 2 for now.[18] The boundary conditions are interesting. Equations (D.15) tell us that

$$\begin{aligned} X^j(x^1, x^2 + 2\pi R_2, t) &= \sigma^3 X^j(x^1, x^2, t)\sigma^3, \\ A^{0,1,2}(x^1, x^2 + 2\pi R_2, t) &= \sigma^3 A^{0,1,2}(x^1, x^2, t)\sigma^3. \end{aligned} \tag{D.18}$$

Combined with equations (D.16) we obtain

$$\begin{aligned} X^j(x^1, -x^2 + 2\pi R_2, t) &= \sigma^2 (X^j(x,t))^T \sigma^2, \\ A^{0,1}(x^1, -x^2 + 2\pi R_2, t) &= -\sigma^2 (A^{0,1}(x^1, x^2, t))^T \sigma^2, \\ A^2(x^1, -x^2 + 2\pi R_2, t) &= \sigma^2 (A^2(x^1, x^2, t))^T \sigma^2. \end{aligned} \tag{D.19}$$

The boundary conditions at the two orientifold planes are, hence, different.[19] For $x^2 = 0$ the matrices satisfy

$$X^j = \sigma^1 (X^j)^T \sigma^1, \qquad \partial_2 X^j = -\sigma^1 (\partial_2 X^j)^T \sigma^1, \tag{D.20}$$

$$A^{0(1)} = -\sigma^1 (A^{0(1)})^T \sigma^1, \qquad \partial_2 A^{0(1)} = \sigma^1 (\partial_2 A^{0(1)})^T \sigma^1, \tag{D.21}$$

$$A^2 = \sigma^1 (A^2)^T \sigma^1, \qquad \partial_2 A^2 = -\sigma^1 (\partial_2 A^2)^T \sigma^1, \tag{D.22}$$

so they are of the form

$$\begin{aligned} A^2, \; X^j &= \begin{pmatrix} N & S \\ \tilde S & N^T \end{pmatrix} \qquad N^+ = N, \; S^T = S, \; \tilde S^T = \tilde S, \; S^+ = \tilde S, \\ A^0, \; A^1 &= \begin{pmatrix} M & R \\ \tilde R & -M^T \end{pmatrix} \qquad M^+ = M, \; R^T = -R, \; \tilde R^T = -\tilde R, \; R^+ = \tilde R. \end{aligned} \tag{D.23}$$

Collecting all these facts together we get that the gauge group in the bulk is $\mathrm{U}(2N)$ while on the $x^2 = 0$ boundary the gauge group is $\mathrm{SO}(2N)$. On this boundary the fields A^2, X^j are in the symmetric representation of the gauge group.

The second boundary $x^2 = \pi R_2$ is substantially different. We write the boundary conditions there in a familiar form

$$\sigma^2 X^j \sigma^2 = (X^j)^T, \qquad \sigma^2 \partial_2 X^j \sigma^2 = -(\partial_2 X^j)^T, \tag{D.24}$$

$$\sigma^2 A^{0,1} \sigma^2 = -(A^{0,1})^T, \qquad \sigma^2 \partial_2 A^{0,1} \sigma^2 = (\partial_2 A^{0,1})^T, \tag{D.25}$$

$$\sigma^2 A^2(x^1, t) \sigma^2 = (A^2(x^1, t))^T, \qquad \sigma^2 \partial_2 A^2 \sigma^2 = -(\partial_2 A^2)^T. \tag{D.26}$$

[18] The Lagrangian (D.17) on the cylinder defines what we mean by $g_{\rm YM}$ so there is no ambiguity.

[19] These are the anticipated O^- and O^+ planes.

Consequently, the gauge group at this boundary is USp$(2N)$. The most general matrices are of the form

$$\begin{aligned} A^2\,,\ X^j &= \begin{pmatrix} N & S \\ \tilde{S} & N^T \end{pmatrix} \qquad && N^+ = N,\ S^T = -S,\ \tilde{S}^T = -\tilde{S},\ S^+ = \tilde{S}, \\ A^0\,,\ A^1 &= \begin{pmatrix} M & R \\ \tilde{R} & -M^T \end{pmatrix} \qquad && M^+ = M,\ R^T = R,\ \tilde{R}^T = \tilde{R},\ R^+ = \tilde{R}. \end{aligned} \tag{D.27}$$

It is easy to verify that this precisely agrees with the action and boundary conditions that we wrote down in the main text for the D2-brane between two orientifold planes, if we only include the disk contributions to this action and not the Möbius strip contributions.

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